# Fully-Homomorphic Encryption

Chethan Kamath

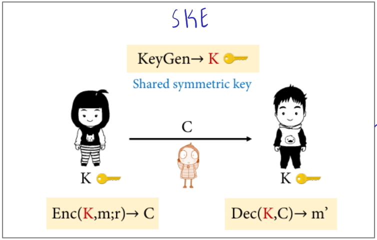
IITB TRUST LAB

DIGITAL : SECURE : RESPONSIBLE

ACM Summer School 2024, 7/Jun/2024

# Recall from Yesterday's Sessions

## SYMMETRIC KEY UNIVERSE



### SKE

KeyGen→ K

Shared symmetric key

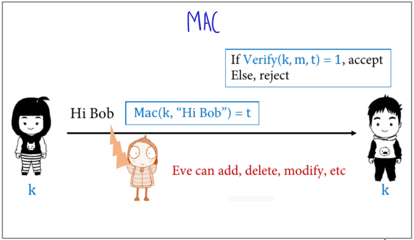
C

K

K

Enc(K,m;r)→ C

Dec(K,C)→ m'



### MAC

If Verify(k, m, t) = 1, accept
Else, reject

Hi Bob

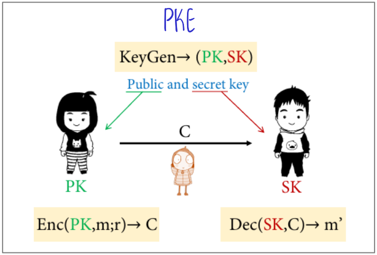
Mac(k, "Hi Bob") = t

Eve can add, delete, modify, etc

k

k

## PUBLIC KEY UNIVERSE



### PKE

KeyGen→ (PK,SK)

Public and secret key

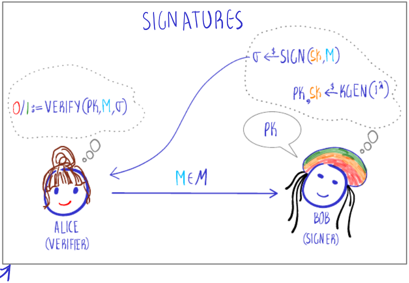
C

PK

SK

Enc(PK,m;r)→ C

Dec(SK,C)→ m'

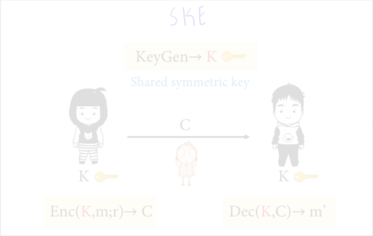


### SIGNATURES

$\sigma \xleftarrow{\$} \mathrm{SIGN}(SK, M)$
$PK, SK \xleftarrow{\$} \mathrm{KGEN}(1^{\lambda})$
$0/1 := \mathrm{VERIFY}(PK, M, \sigma)$
PK
$M \in \mathcal{M}$
ALICE
(VERIFIER)
BOB
(SIGNER)

# Recall from Yesterday's Sessions

## SYMMETRIC KEY UNIVERSE



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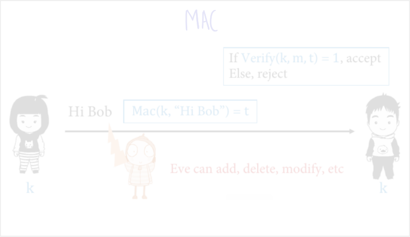

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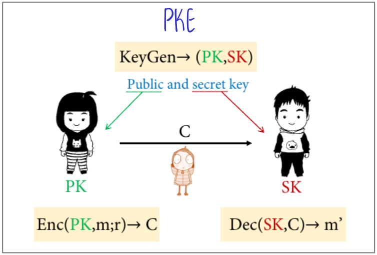

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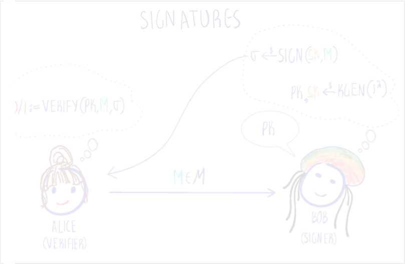

SIGNATURES

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$PK, SK \xleftarrow{\$} \mathrm{KGEN}(1^\lambda)$
$0/1 := \mathrm{VERIFY}(PK,M,\sigma)$
PK
M
ALICE (VERIFIER)
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# Recall from Yesterday's Sessions

## SYMMETRIC KEY UNIVERSE

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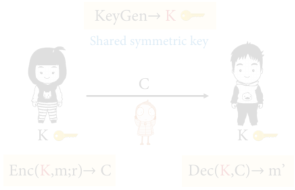

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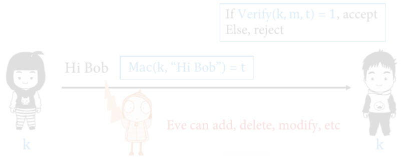
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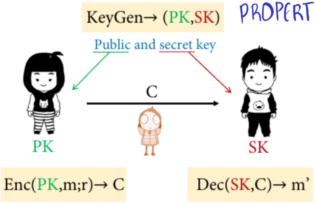
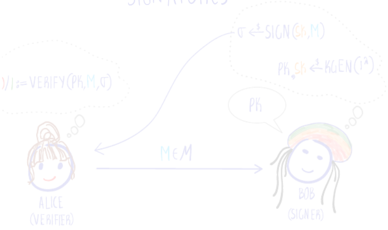
## PUBLIC KEY UNIVERSE

### PKE W/ HOMOMORPHIC PROPERTIES



### SIGNATURES



# Plan for this Session

Homomorphic Encryption

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Fully-Homomorphic Encryption (FHE)

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Learning with Errors (LWE)

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Wrapping Up

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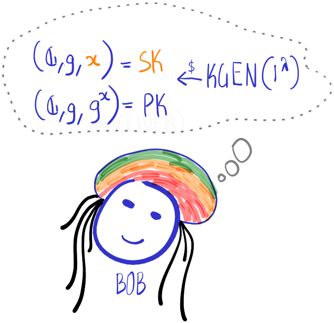
Wrapping Up

# Example 1: Elgamal Encryption

- What happens when we multiply ciphertexts?
- Is it possible to compute sum of plaintexts?

# Example 1: Elgamal Encryption



$$(\mathbb{G}, g, x) = SK \xleftarrow{\$} KGEN(1^\lambda)$$

$$(\mathbb{G}, g, g^x) = PK$$

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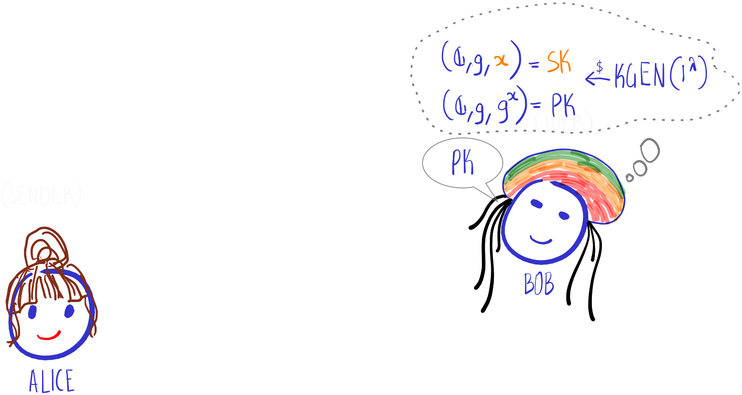
# Example 1: Elgamal Encryption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(RECEIVER)
BOB
(SENDER)
ALICE

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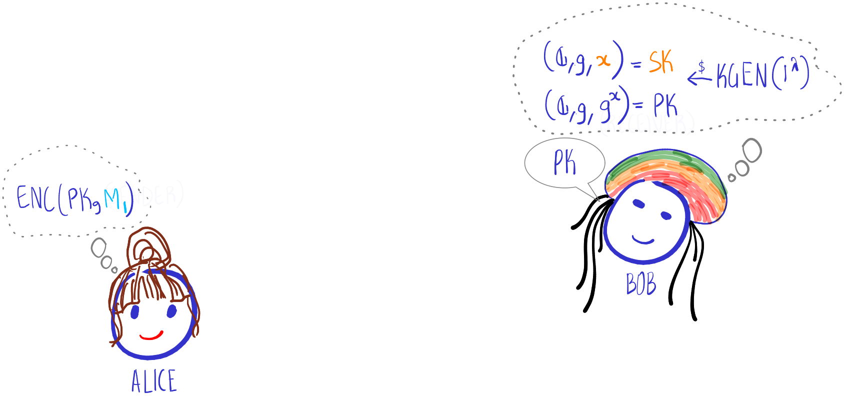
$(\mathbb{G}, g, x) = SK \overset{\$}{\leftarrow} KGEN(1^\lambda)$
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BOB
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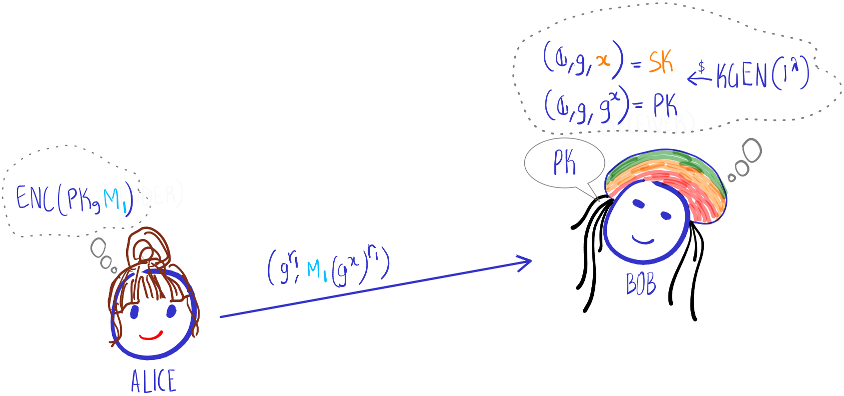
$(\mathbb{G}, g, x) = SK \xleftarrow{\$} KGEN(1^\lambda)$
$(\mathbb{G}, g, g^x) = PK$
PK
$ENC(PK, M_1)$
$(g^{r_1}, M_1 (g^x)^{r_1})$
ALICE
BOB

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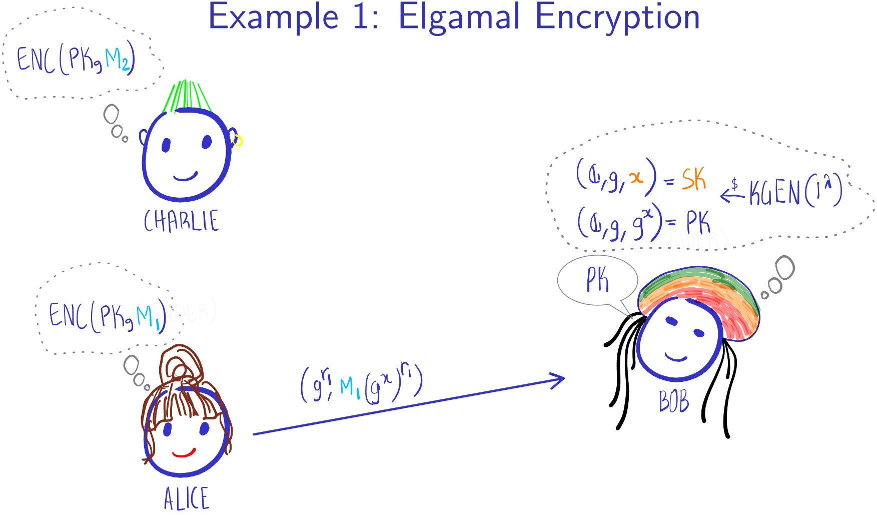
$\mathrm{ENC}(PK_g, M_2)$
CHARLIE
$(\mathbb{G}, g, x) = SK \xleftarrow{\$} \mathrm{KGEN}(1^\lambda)$
$(\mathbb{G}, g, g^x) = PK$
(RECEIVER)
PK
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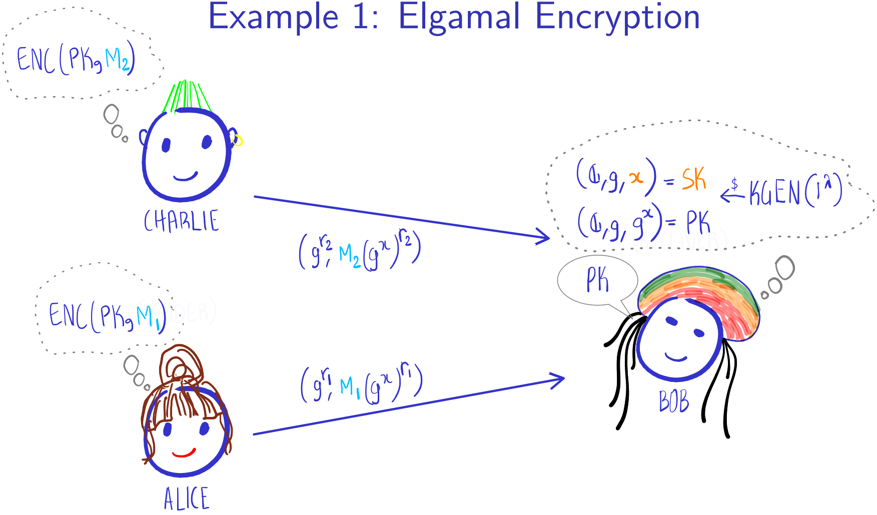
$\mathrm{ENC}(PK, M_2)$
CHARLIE
$(g^{r_2}, M_2 (g^x)^{r_2})$
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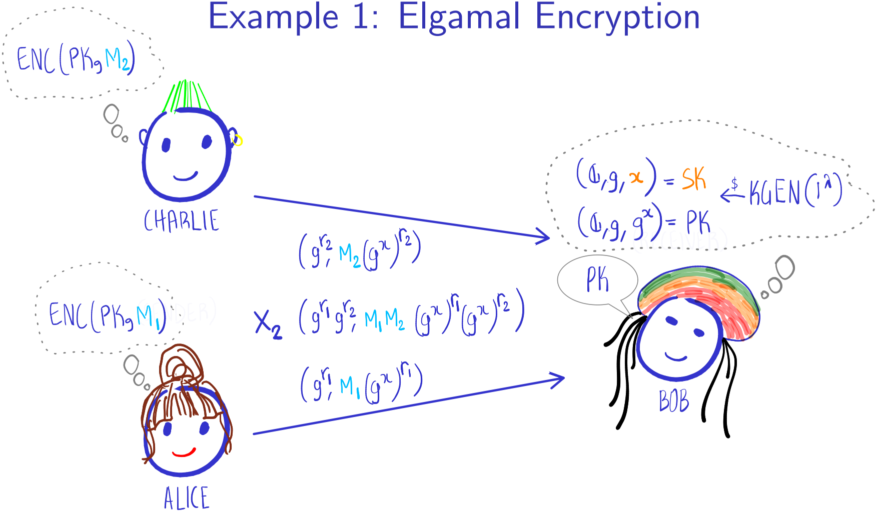
$ENC(PK, M_2)$
CHARLIE
$(g^{r_2}, M_2 (g^x)^{r_2})$
$X_2$ $(g^{r_1} g^{r_2}, M_1 M_2 (g^x)^{r_1} (g^x)^{r_2})$
$(g^{r_1}, M_1 (g^x)^{r_1})$
$ENC(PK, M_1)$
ALICE
$(\mathbb{G}, g, x) = SK \xleftarrow{\$} KGEN(1^\lambda)$
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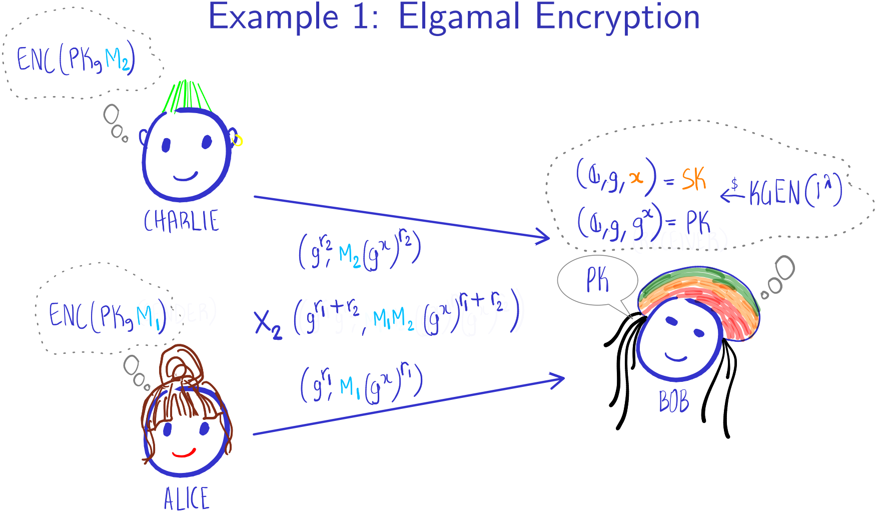
$ENC(PK_g, M_2)$
CHARLIE
$ENC(PK_g, M_1)$
ALICE
$(g^{r_2}, M_2(g^x)^{r_2})$
$X_2 \; (g^{r_1+r_2}, M_1M_2 (g^x)^{r_1+r_2})$
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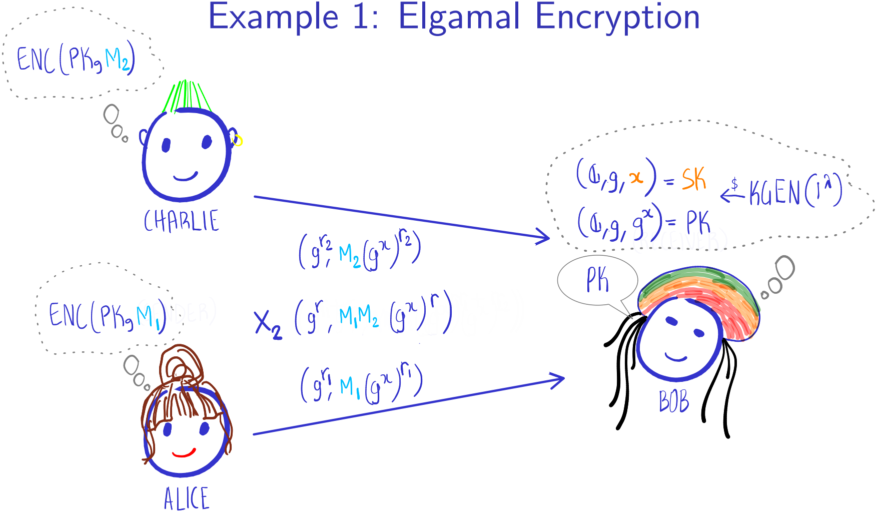
ENC(PK, $M_2$)
CHARLIE
$(\mathbb{G}, g, x) = SK \xleftarrow{\$} KGEN(1^\lambda)$
$(\mathbb{G}, g, g^x) = PK$
PK
BOB
$(g^{r_2}, M_2 (g^x)^{r_2})$
$\times_2$ $(g^r, M_1 M_2 (g^x)^r)$
$(g^{r_1}, M_1 (g^x)^{r_1})$
ENC(PK, $M_1$)
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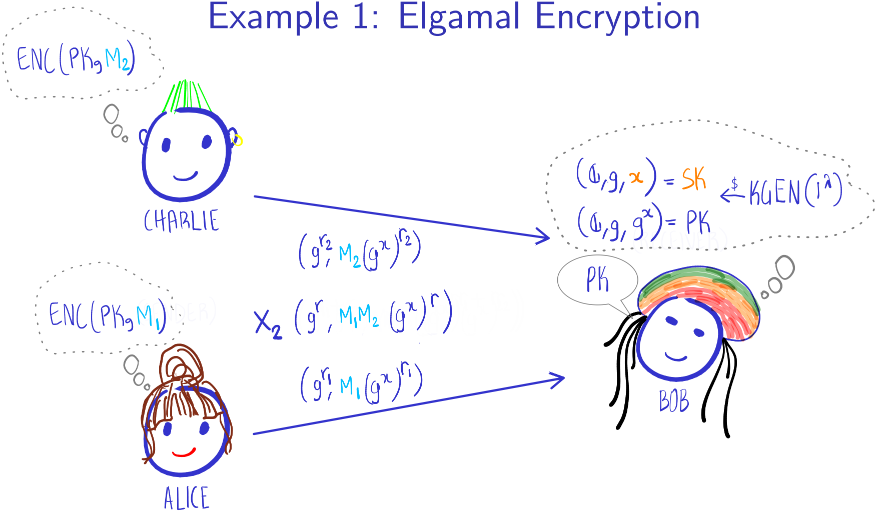
$\mathrm{ENC}(PK, M_2)$
CHARLIE
$(g^{r_2}, M_2 (g^x)^{r_2})$
$\mathrm{ENC}(PK, M_1)$
$X_2 \ (g^r, M_1 M_2 (g^x)^r)$
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$(\mathbb{G}, g, x) = SK \xleftarrow{\$} \mathrm{KGEN}(1^\lambda)$
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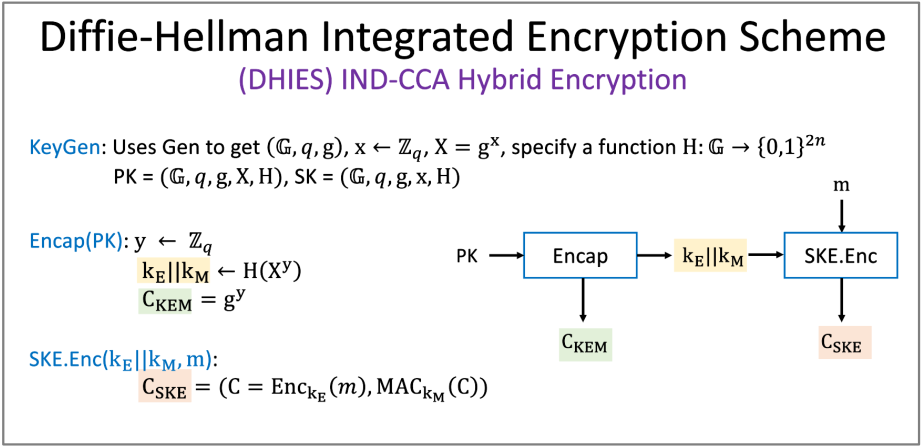
# What about DHIES?



## Diffie-Hellman Integrated Encryption Scheme

### (DHIES) IND-CCA Hybrid Encryption

KeyGen: Uses Gen to get $(\mathbb{G}, q, \mathrm{g})$, $\mathrm{x} \leftarrow \mathbb{Z}_q$, $\mathrm{X} = \mathrm{g}^{\mathrm{x}}$, specify a function $\mathrm{H}: \mathbb{G} \rightarrow \{0,1\}^{2n}$

PK = $(\mathbb{G}, q, \mathrm{g}, \mathrm{X}, \mathrm{H})$, SK = $(\mathbb{G}, q, \mathrm{g}, \mathrm{x}, \mathrm{H})$

Encap(PK): $\mathrm{y} \leftarrow \mathbb{Z}_q$

$\mathrm{k_E}||\mathrm{k_M} \leftarrow \mathrm{H}(\mathrm{X}^{\mathrm{y}})$

$\mathrm{C_{KEM}} = \mathrm{g}^{\mathrm{y}}$

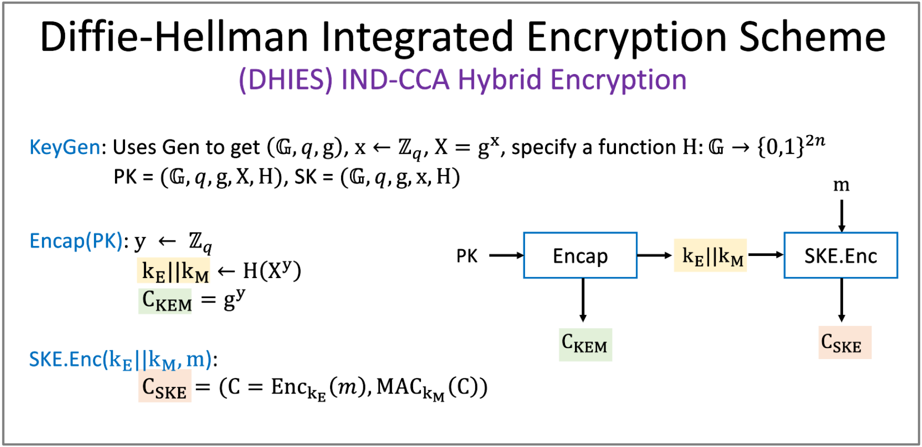
SKE.Enc($\mathrm{k_E}||\mathrm{k_M}, \mathrm{m}$):

$\mathrm{C_{SKE}} = (\mathrm{C} = \mathrm{Enc}_{\mathrm{k_E}}(m), \mathrm{MAC}_{\mathrm{k_M}}(\mathrm{C}))$

Exercise 1

*What happens when we (say) XOR ciphertexts?*

# What about DHIES?



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PK = $(\mathbb{G}, q, g, X, H)$, SK = $(\mathbb{G}, q, g, x, H)$

Encap(PK): $y \leftarrow \mathbb{Z}_q$

$k_E || k_M \leftarrow H(X^y)$

$C_{KEM} = g^y$

SKE.Enc($k_E || k_M$, m):

$C_{SKE} = (C = \mathrm{Enc}_{k_E}(m), \mathrm{MAC}_{k_M}(C))$

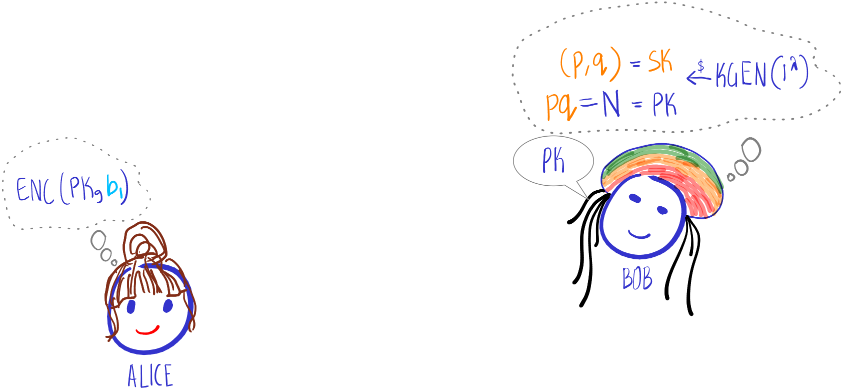
**Exercise 1**

*What happens when we (say) XOR ciphertexts?*

# Example 2: Goldwasser-Micali Bit Encryption

- What happens when we multiply ciphertexts?
- Is it possible compute product of plaintexts (modulo 2)?

# Example 2: Goldwasser-Micali Bit Encryption



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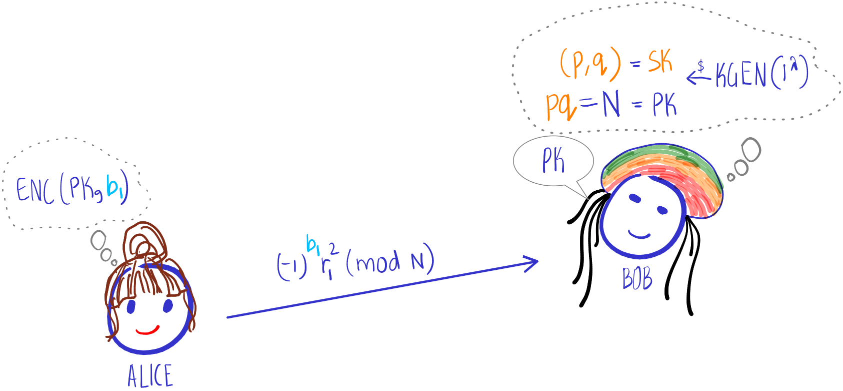

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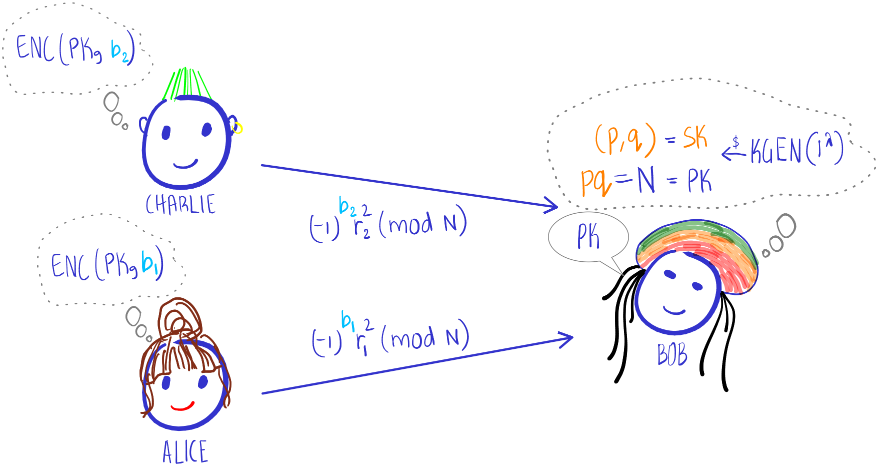
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CHARLIE
$ENC(PK_g, b_1)$
ALICE
$(p,q) = SK \overset{\$}{\leftarrow} KGEN(1^\lambda)$
$pq = N = PK$
PK
BOB
$(-1)^{b_2} r_2^2 \pmod N$
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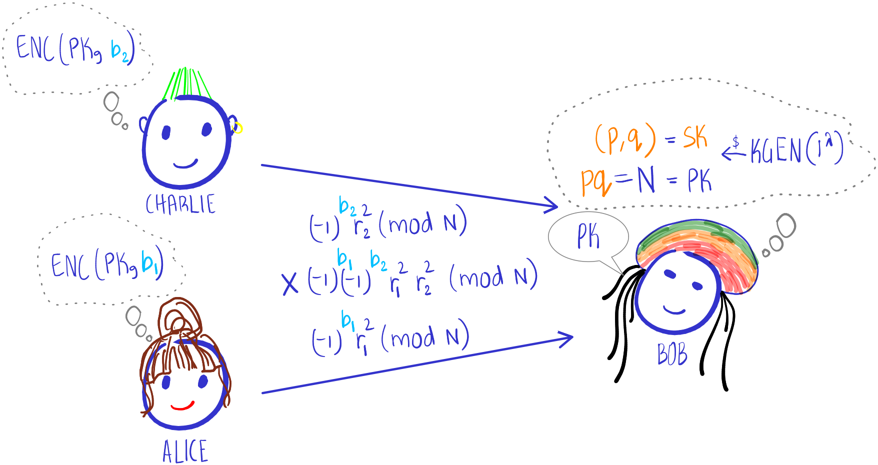
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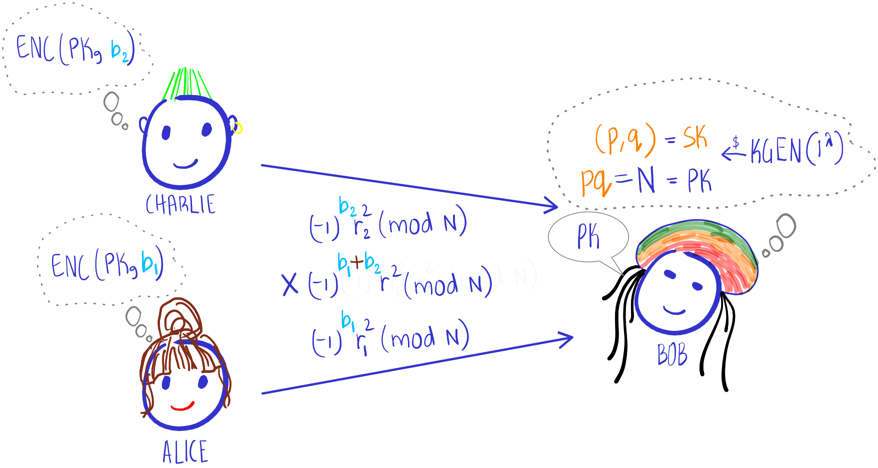
$\mathrm{ENC}(PK_B, b_2)$
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# Plan for this Session

Homomorphic Encryption

Fully-Homomorphic Encryption (FHE)

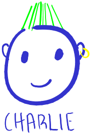
Learning with Errors (LWE)

Gentry-Sahai-Waters FHE from LWE

Wrapping Up

# Defining Homomorphic Encryption

- Public-key encryption with additional *evaluation* algorithm
  - Four-tuple of algorithms: $(\texttt{KGEN}, \texttt{ENC}, \texttt{DEC}, \texttt{EVAL})$



- FHE supports evaluation of *arbitrary* functions $F$
- Levelled FHE supports function of depth $L$

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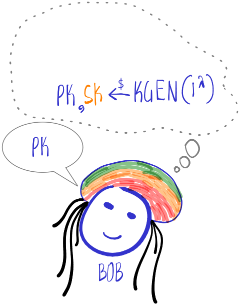

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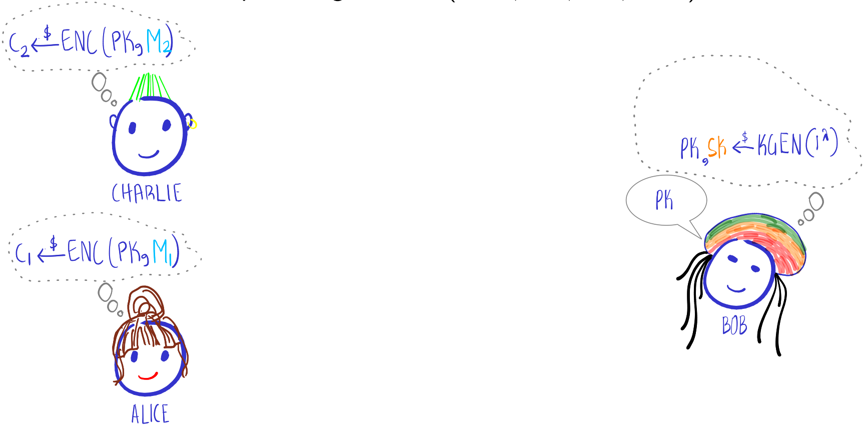
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CHARLIE
$C_1 \xleftarrow{\$} \mathrm{ENC}(PK, M_1)$
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$PK, SK \xleftarrow{\$} \mathrm{KGEN}(1^\lambda)$
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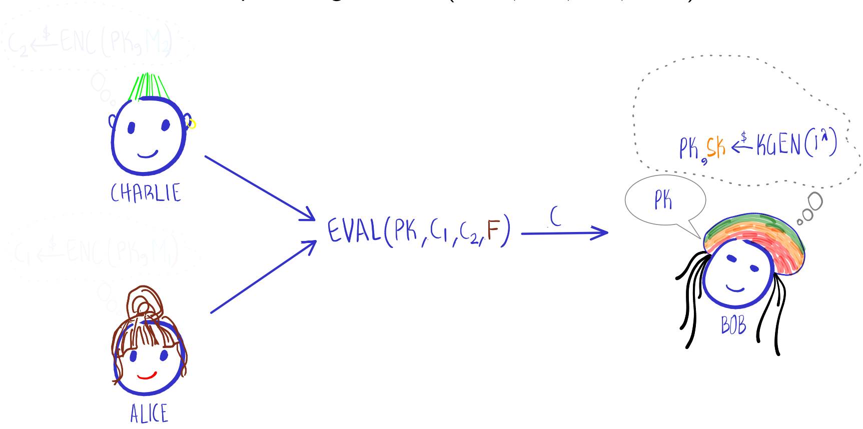

$C_2 \xleftarrow{\$} ENC(PK, M_2)$
CHARLIE
$C_1 \xleftarrow{\$} ENC(PK, M_1)$
ALICE
$EVAL(PK, C_1, C_2, F)$
$C$
$PK, SK \xleftarrow{\$} KGEN(1^\lambda)$
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BOB

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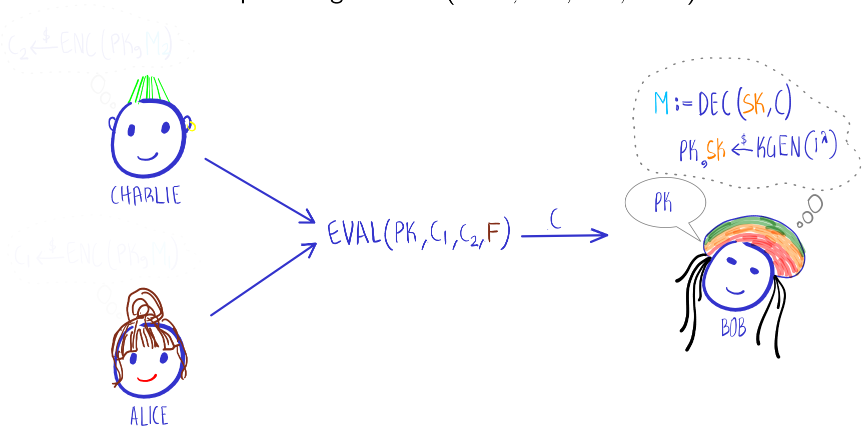

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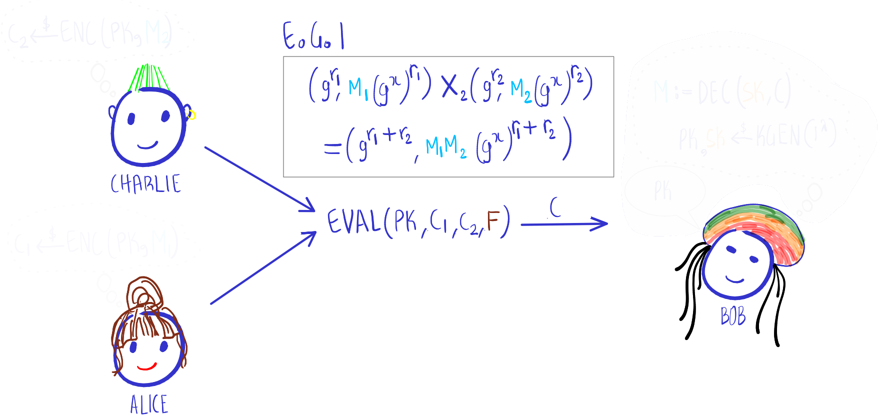

EoGol

$$\left(g^{r_1}, M_1 (g^x)^{r_1}\right) \times_2 \left(g^{r_2}, M_2 (g^x)^{r_2}\right)$$
$$= \left(g^{r_1+r_2}, M_1 M_2 (g^x)^{r_1+r_2}\right)$$

$\mathrm{EVAL}(PK, C_1, C_2, F) \xrightarrow{C}$

$C_2 \xleftarrow{\$} \mathrm{ENC}(PK, M_2)$
CHARLIE
$C_1 \xleftarrow{\$} \mathrm{ENC}(PK, M_1)$
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$M := \mathrm{DEC}(SK, C)$
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PK
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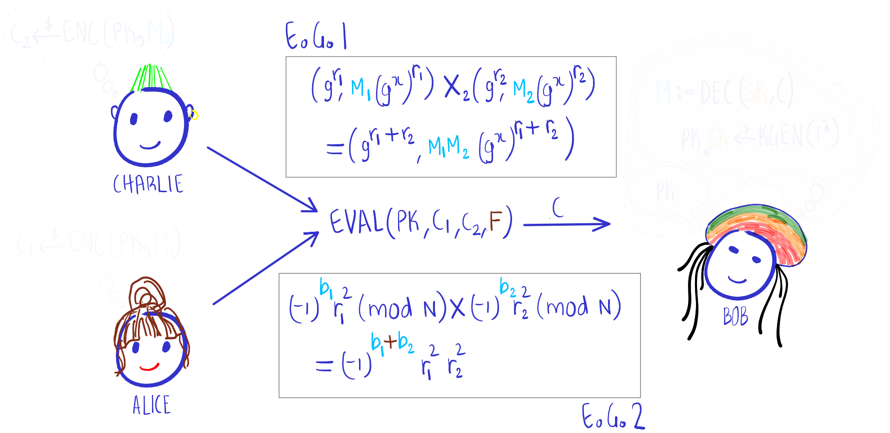
- FHE supports evaluation of *arbitrary* functions $F$
- Levelled FHE supports function of depth $L$

# Defining Homomorphic Encryption

- Public-key encryption with additional *evaluation* algorithm
    - Four-tuple of algorithms: $(\texttt{KGEN}, \texttt{ENC}, \texttt{DEC}, \texttt{EVAL})$



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E.G.1

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$$\mathrm{EVAL}(PK, C_1, C_2, F) \xrightarrow{C}$$

$$(-1)^{b_1} r_1^2 \pmod N \times (-1)^{b_2} r_2^2 \pmod N$$
$$= (-1)^{b_1+b_2} r_1^2 r_2^2$$

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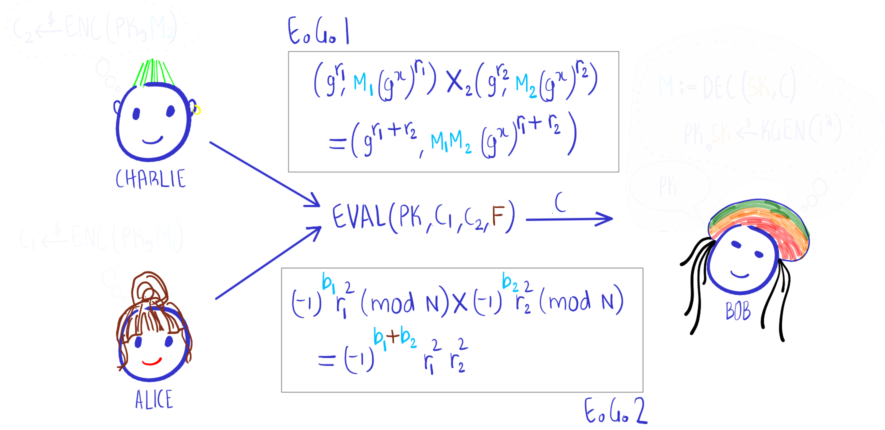

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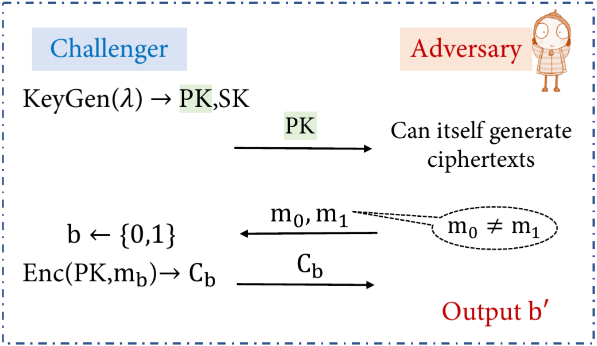
# Security Model: IND-CPA for PKE



Challenger

Adversary

KeyGen($\lambda$) → PK,SK

PK

Can itself generate ciphertexts

$m_0, m_1$

$m_0 \neq m_1$

$b \leftarrow \{0,1\}$

Enc(PK,$m_b$)→ $C_b$

$C_b$

Output $b'$

IND-CPA Security (Chosen Plaintext Attack)

$$\forall \text{ PPT adversaries, } \Pr[b' = b] \leq \frac{1}{2} + \mathrm{negl}(\lambda)$$

Exercise 2 (IND-CCA)

*Can FHE be IND-CCA secure?*

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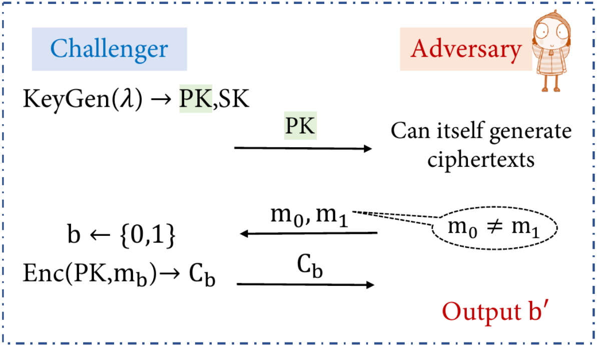

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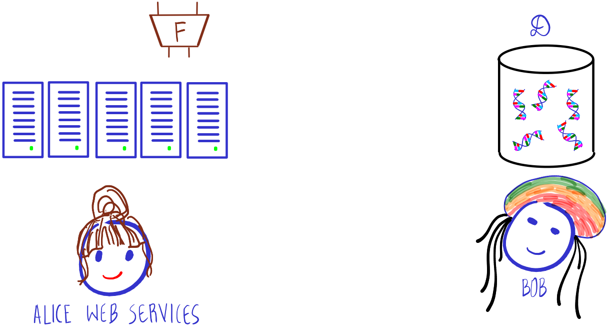

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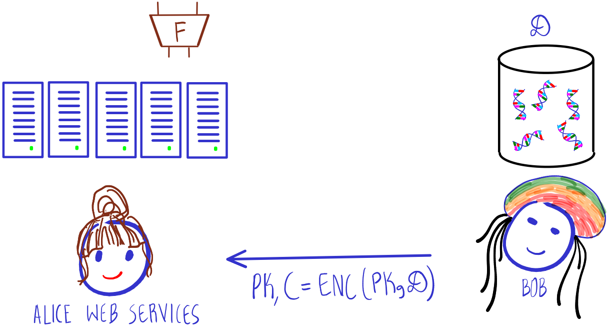

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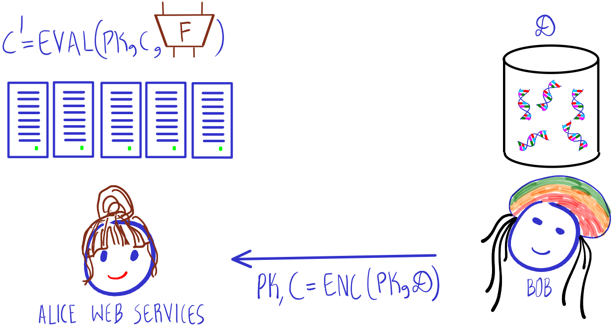

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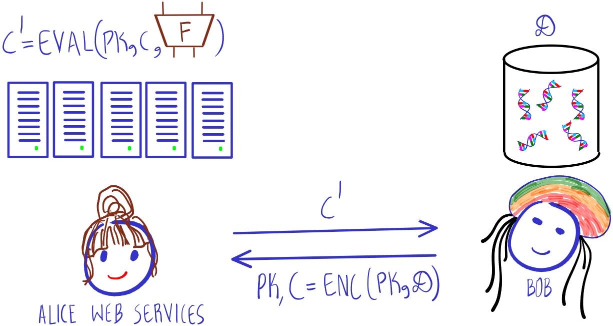

# Plan for this Session

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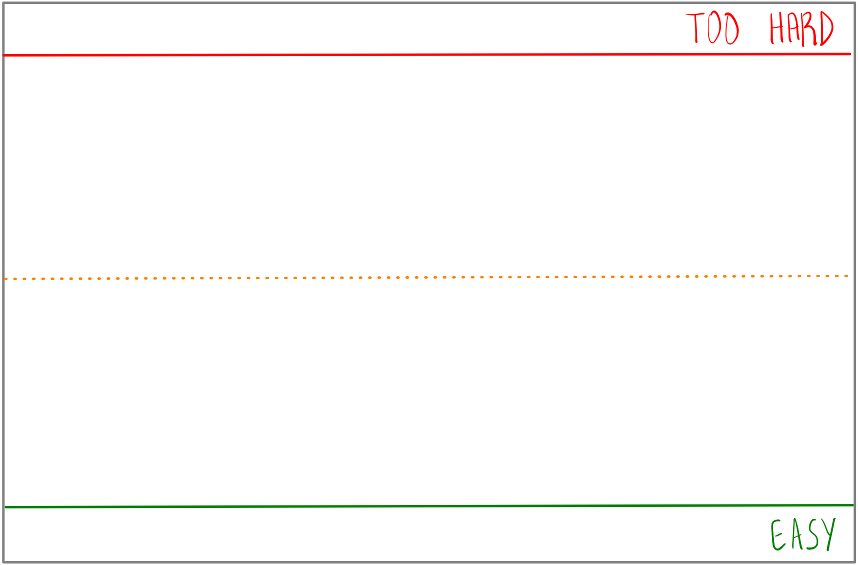
# Cryptography Landscape



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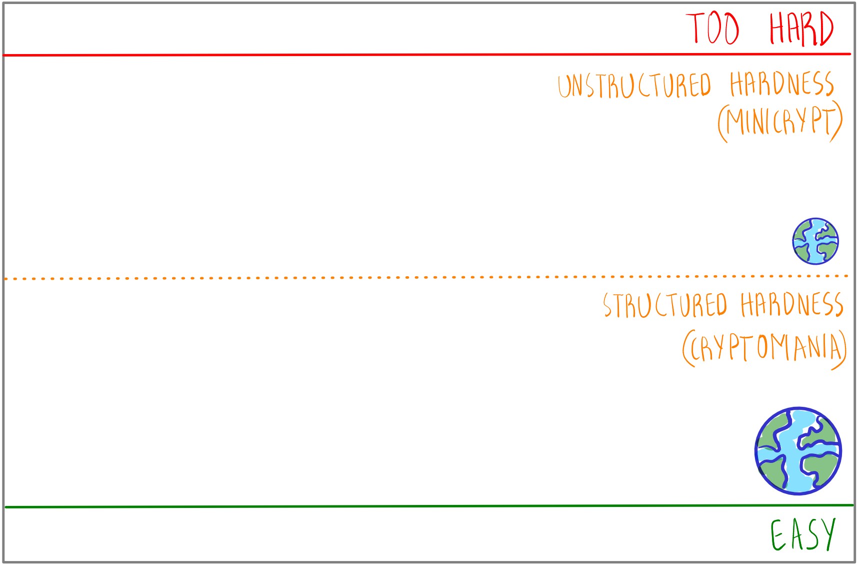

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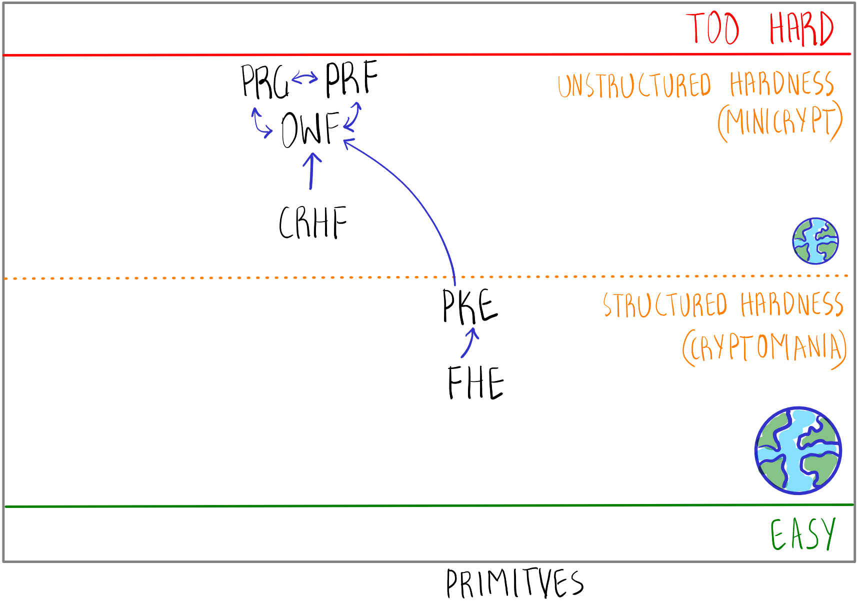

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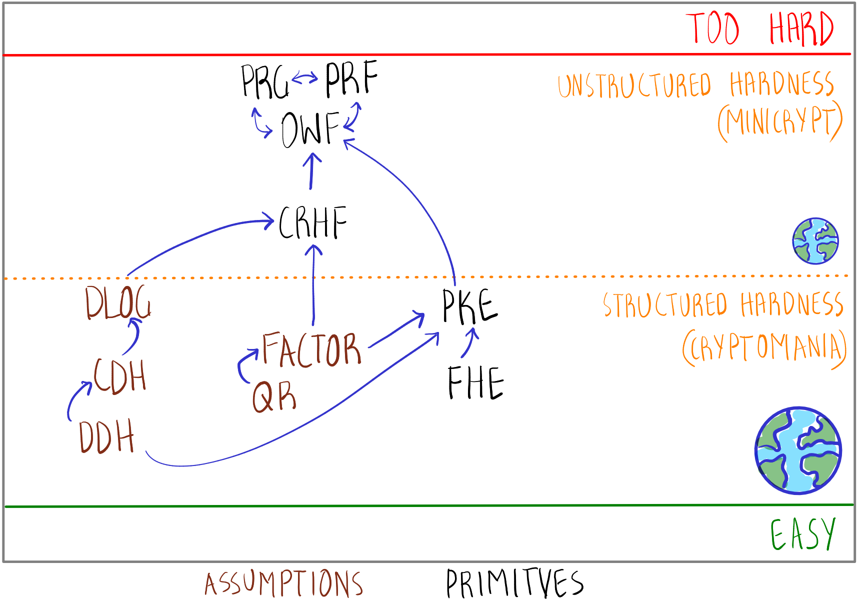

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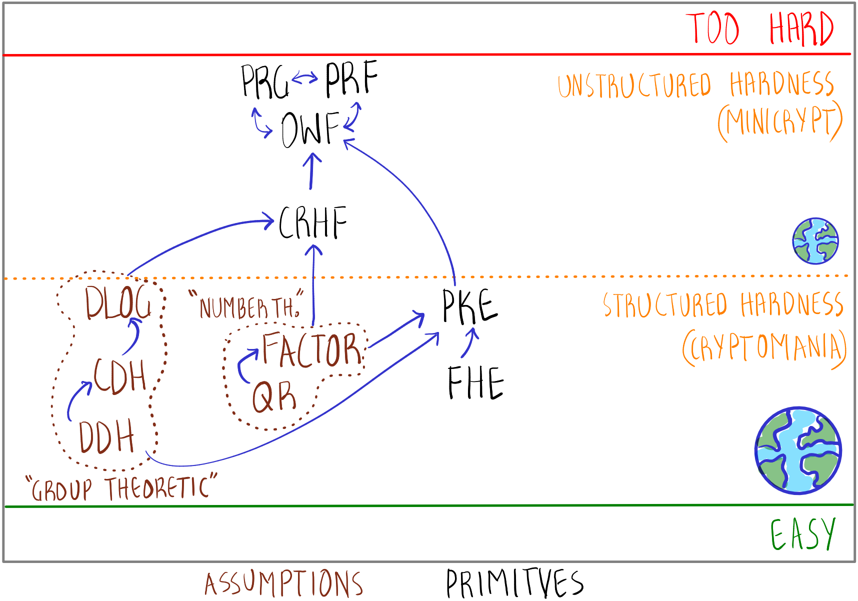

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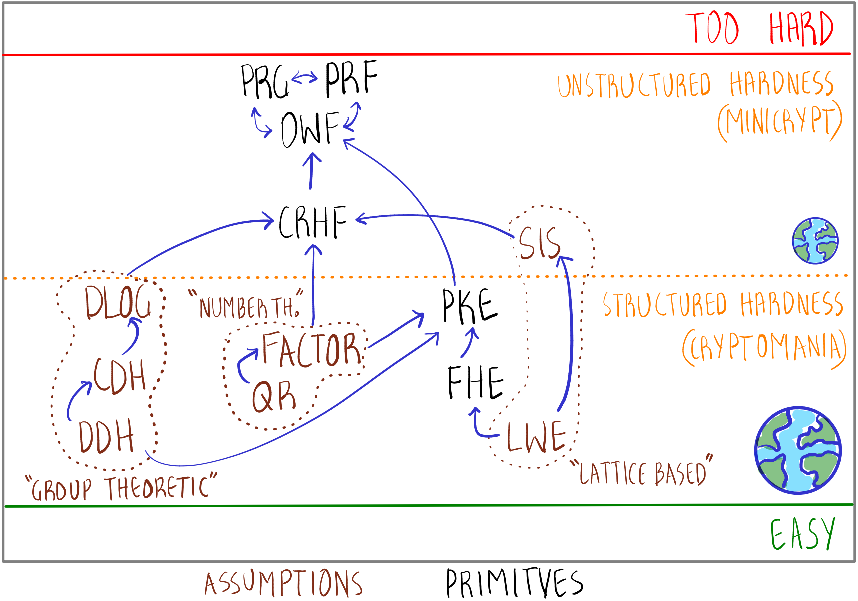

# LWE: Solving "Noisy" Linear Equations is Hard

$$\begin{array}{c|ccccc|}
 & & & m & & \\
\hline
 & a_{11} & & \cdots & & a_{m1} \\
 & & \ddots & & & \\
n & \vdots & & a_{ij} & & \vdots \\
 & & & & \ddots & \\
 & a_{n1} & & \cdots & & a_{mn} \\
\hline
\end{array}$$

- Search vs decision LWE

- Solving LWE is at least as hard as solving certain lattice problems in the *worst case*

# LWE: Solving "Noisy" Linear Equations is Hard

$$\begin{array}{ccc} & m \rightarrow \approx n\cdot\log(q) & \\ \approx \text{SECURITY PARAM.} \leftarrow n & \boxed{\begin{matrix} a_{11} & \cdots & a_{m1} \\ \vdots & a_{ij} & \vdots \\ a_{n1} & \cdots & a_{mn}\end{matrix}} & a_{ij} \in \mathbb{Z}_q \rightarrow \text{SMALL PRIME} \end{array}$$

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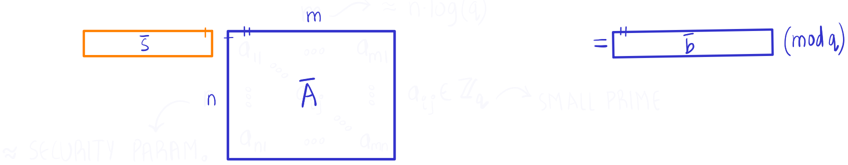

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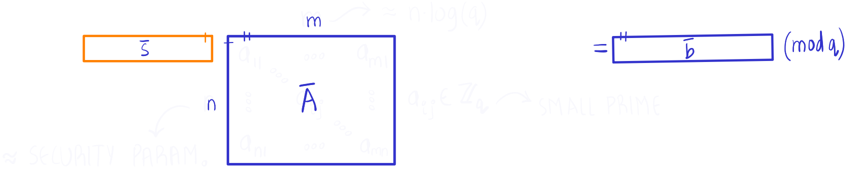

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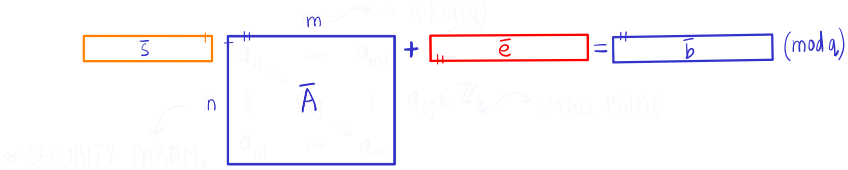

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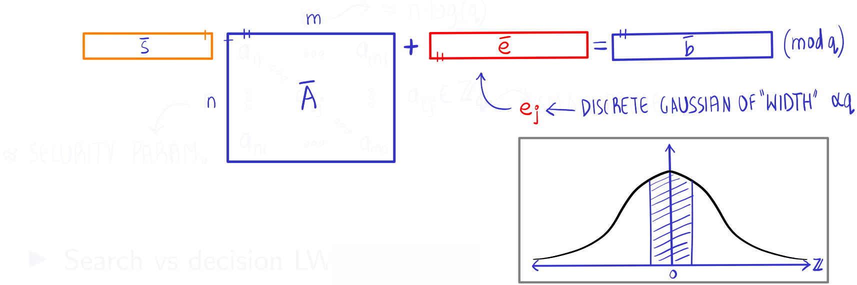

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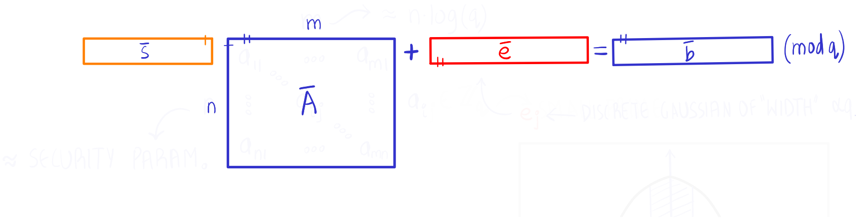

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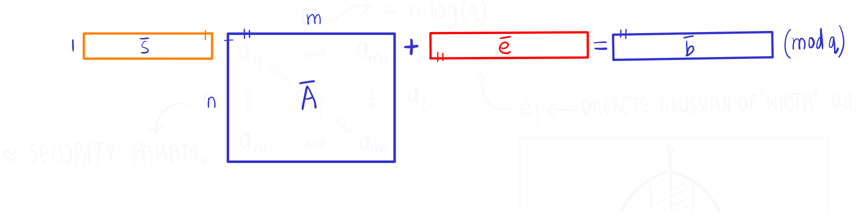

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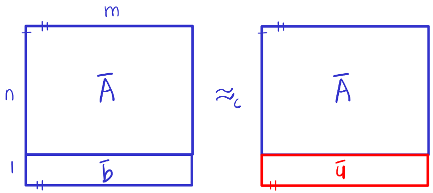

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- What happens when you add two ciphertexts?

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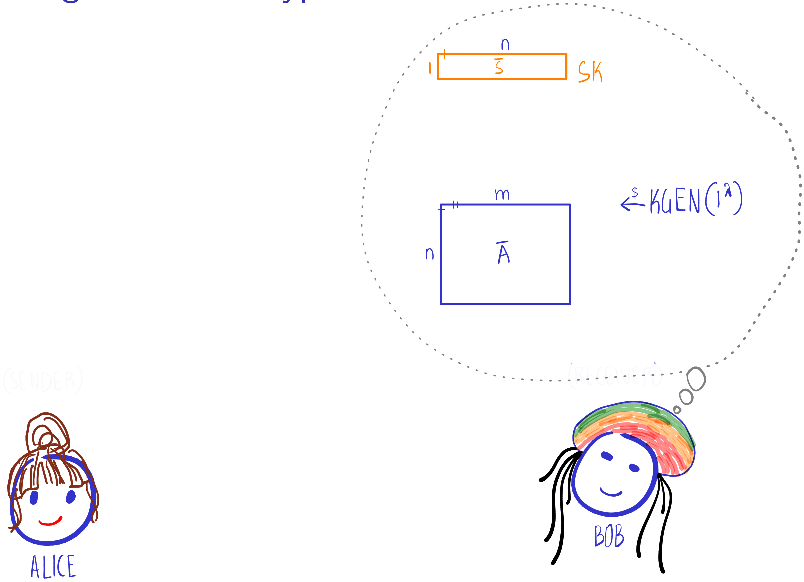

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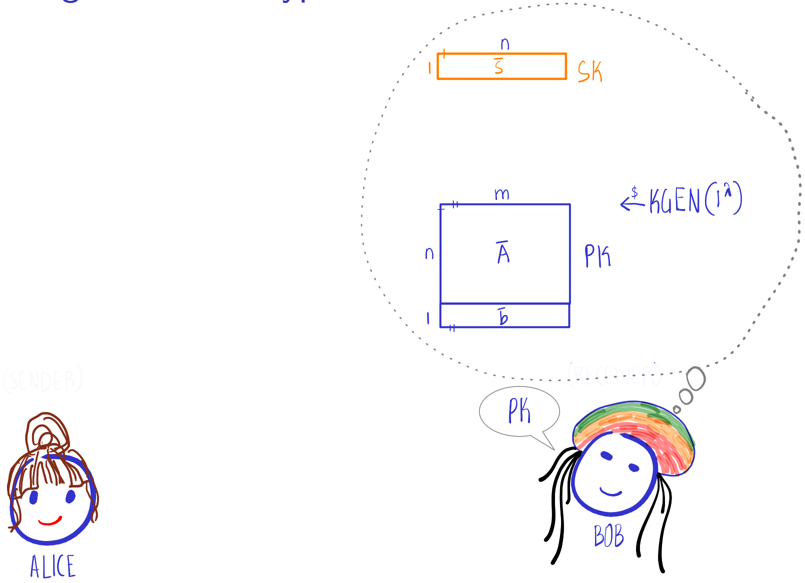

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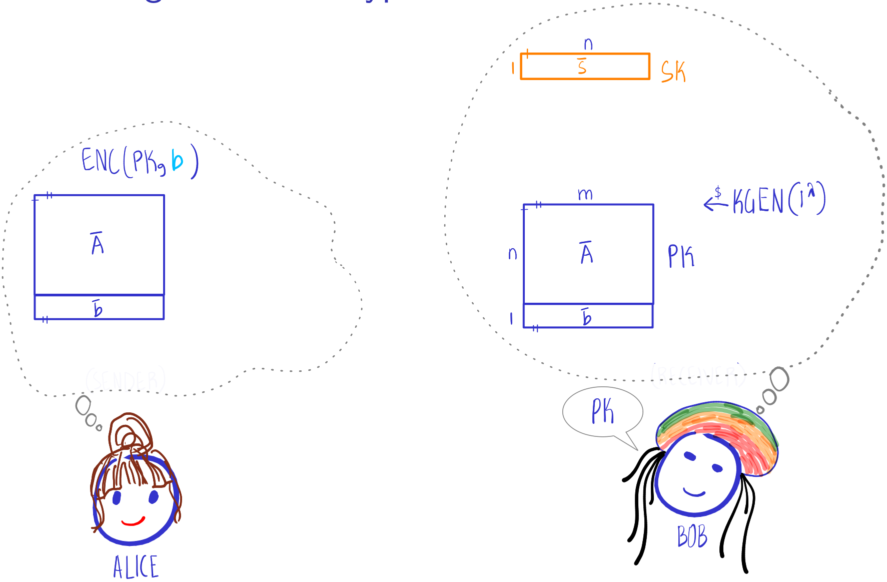

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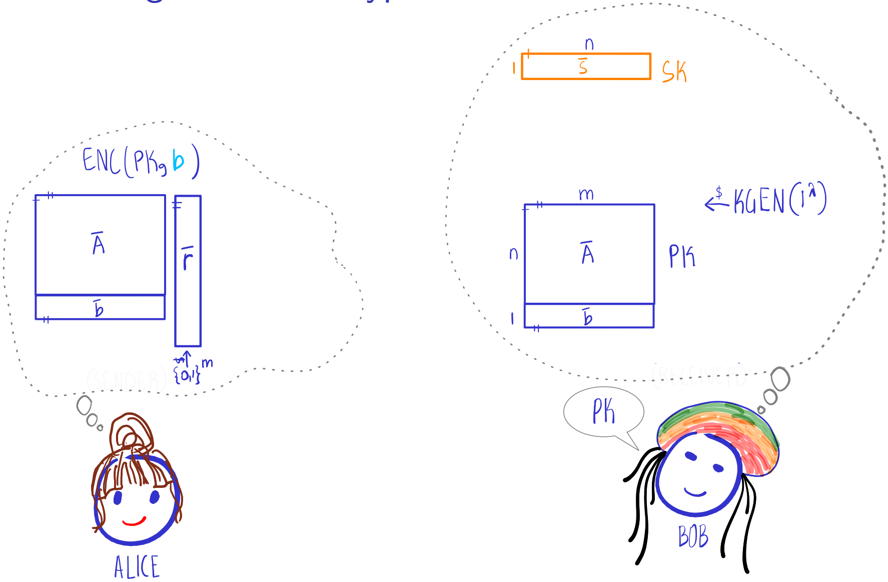

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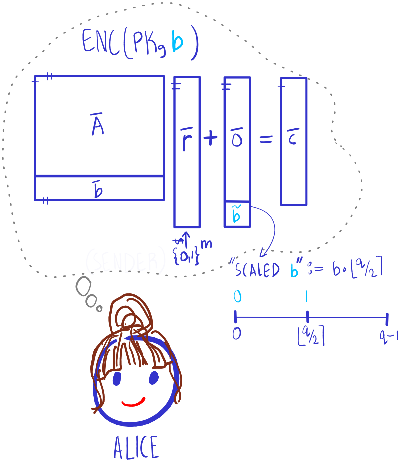

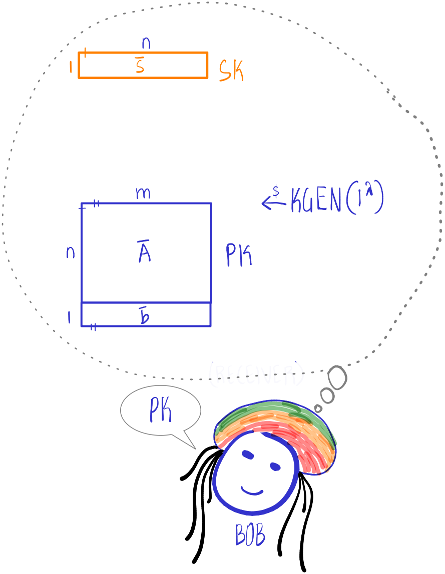

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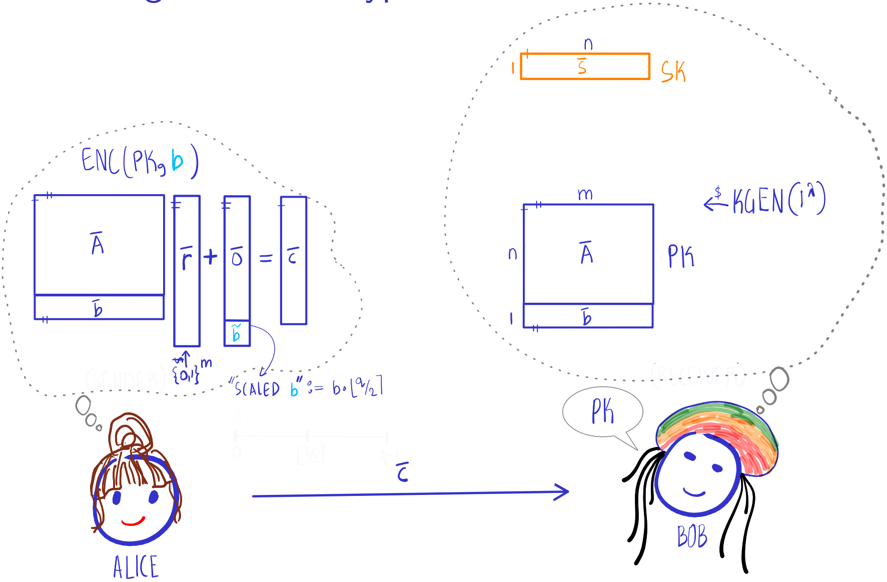

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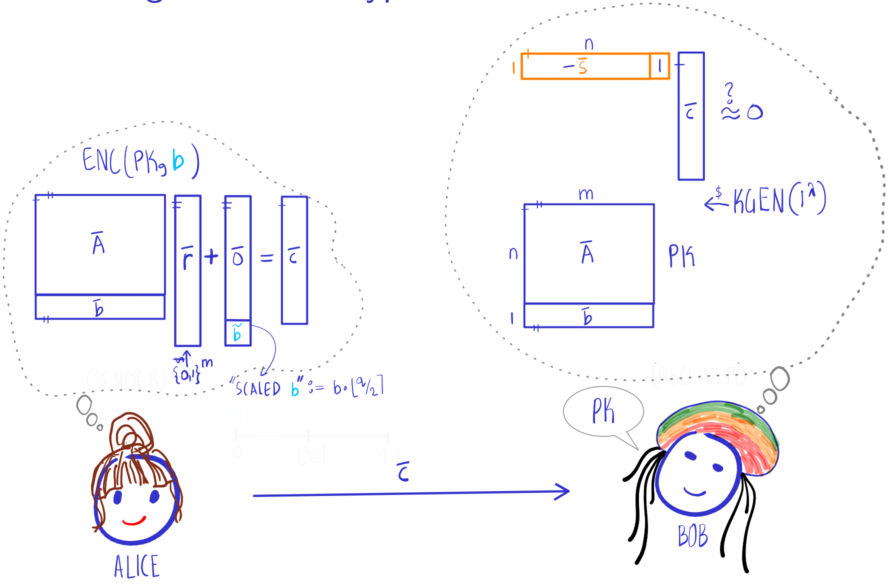
ENC(PK, b)
$\bar{A}$
$\bar{b}$
$\bar{r}$ + $\bar{0}$ = $\bar{c}$
$\tilde{b}$
$\{0,1\}^m$
"SCALED b" := $b \cdot \lfloor q/2 \rceil$
(SENDER)
ALICE
$\bar{c}$
$-\bar{s}$ | 1
$\bar{c}$ $\overset{?}{\approx} 0$
n
1
m
n
$\bar{A}$
1
$\bar{b}$
PK
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(RECEIVER)
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BOB

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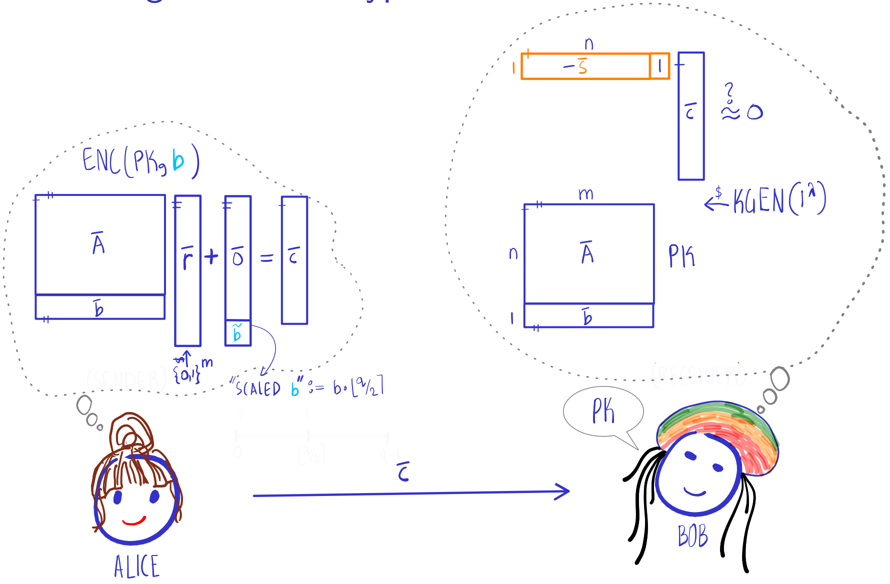

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*Prove security formally.*

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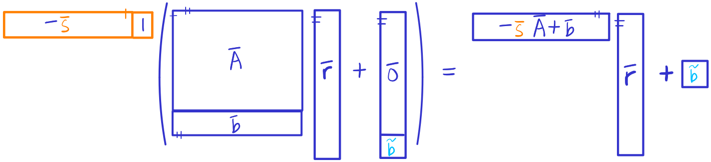

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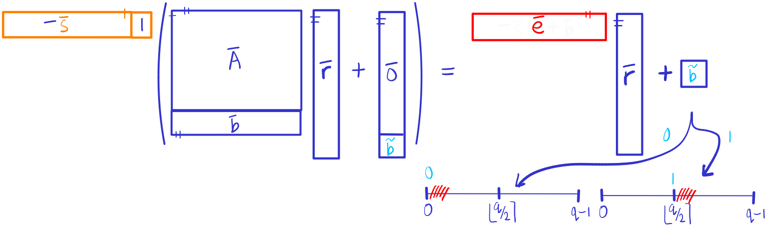

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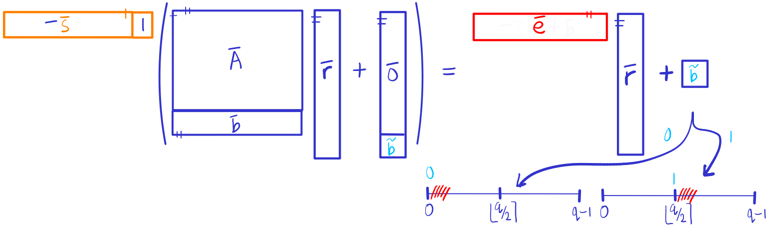

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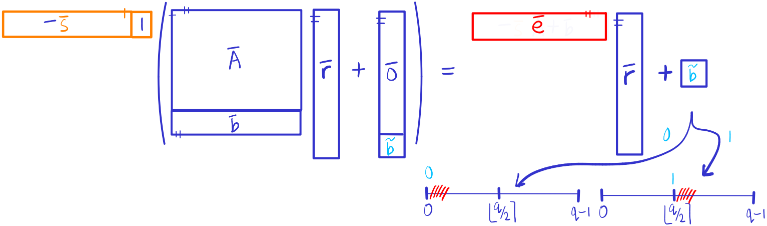

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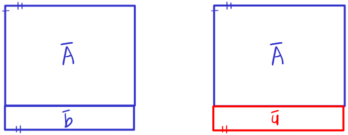

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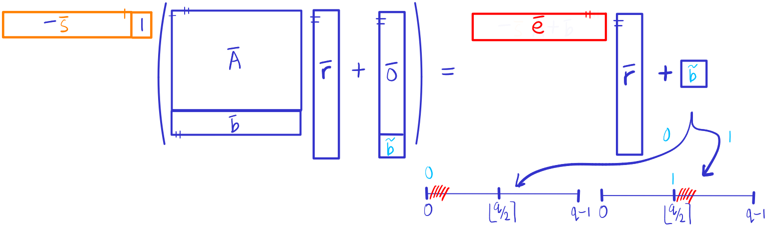

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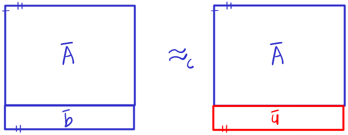

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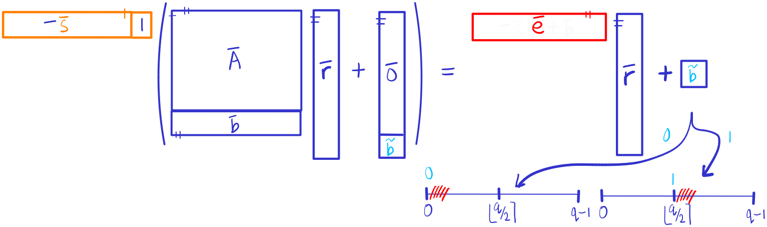

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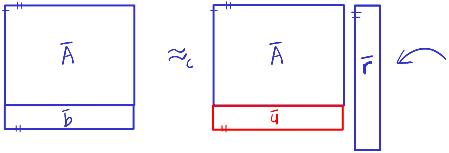

STATISTICALLY HIDES b

"LEFTOVER HASH LEMMA"

### Exercise 3 (Security of Regev's Encryption)

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# Plan for this Session

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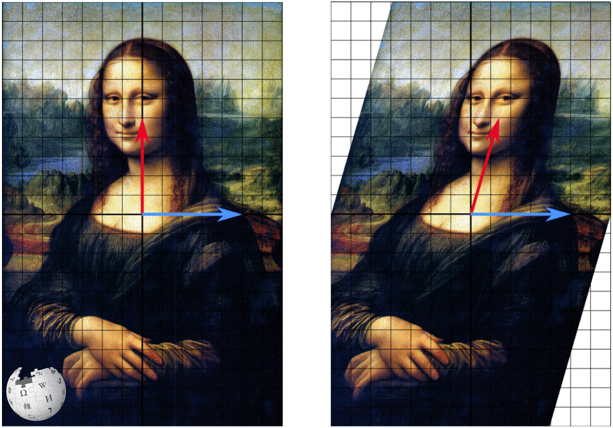
# Let's Recall Eigenvectors



**Definition 1**

A (left) eigenvector of a square matrix $\bar{C}$ is a vector $\bar{v}$ such that $\bar{v}\bar{C} = \mu\bar{v}$ for some scalar $\mu$, which is the eigenvalue.

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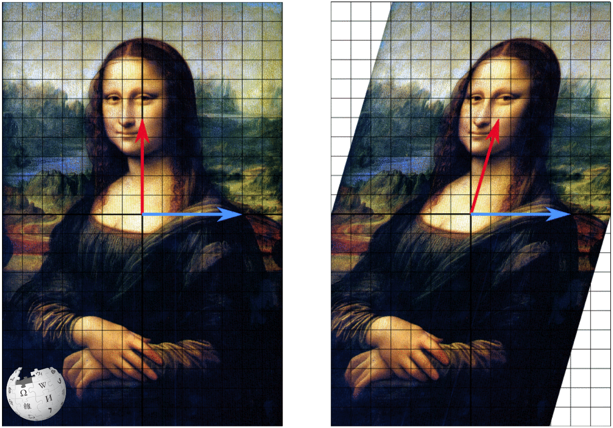

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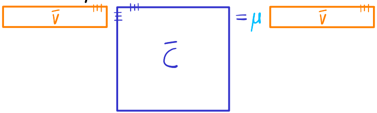

- Do we have an FHE?

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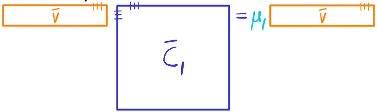

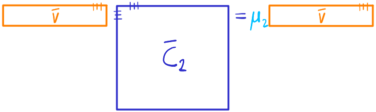

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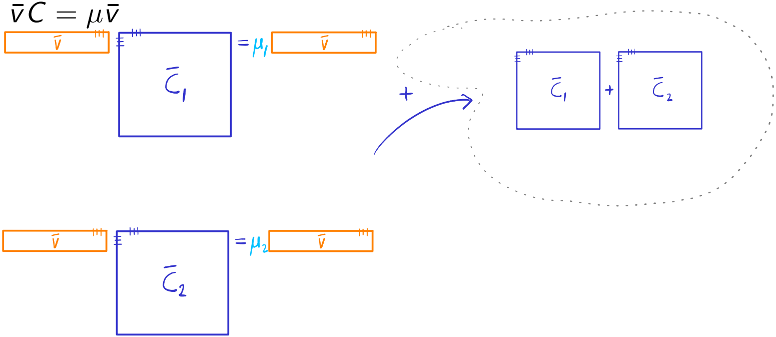

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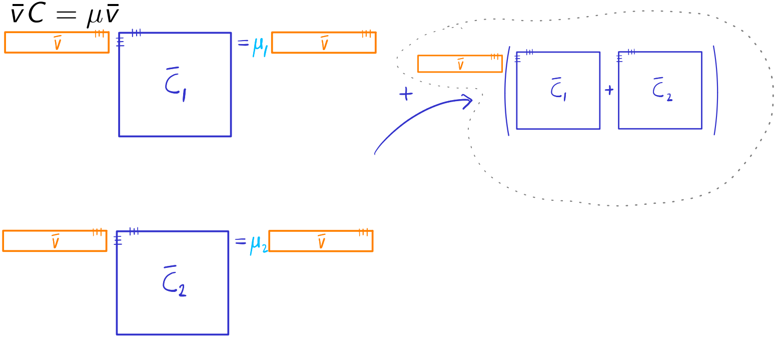

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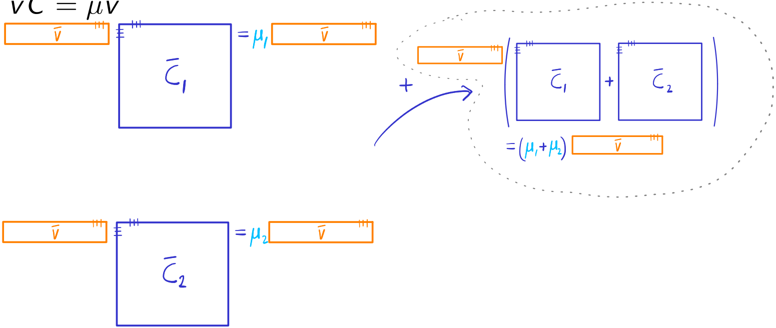

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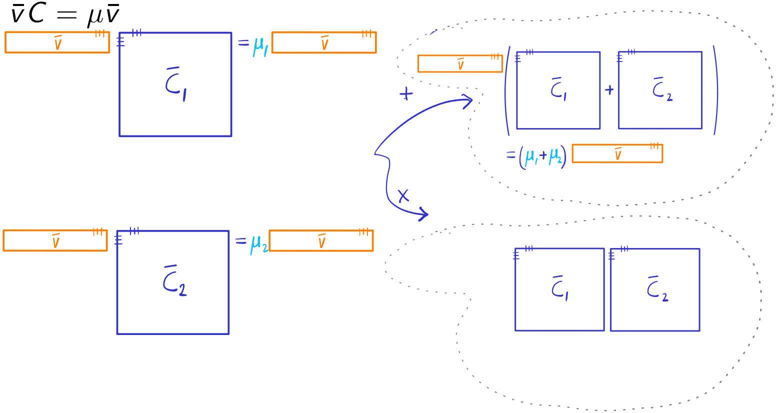

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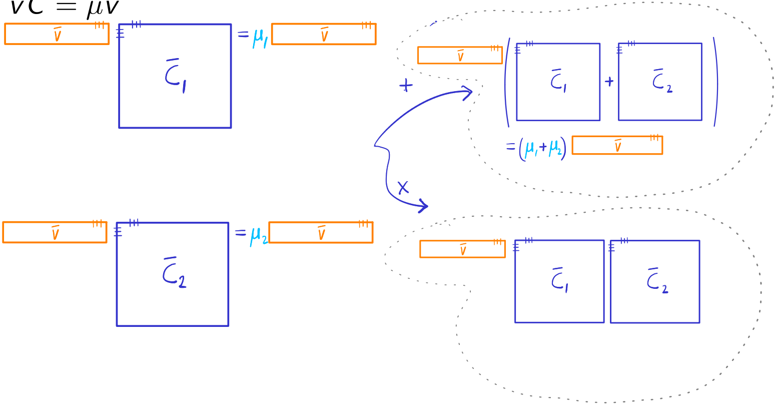

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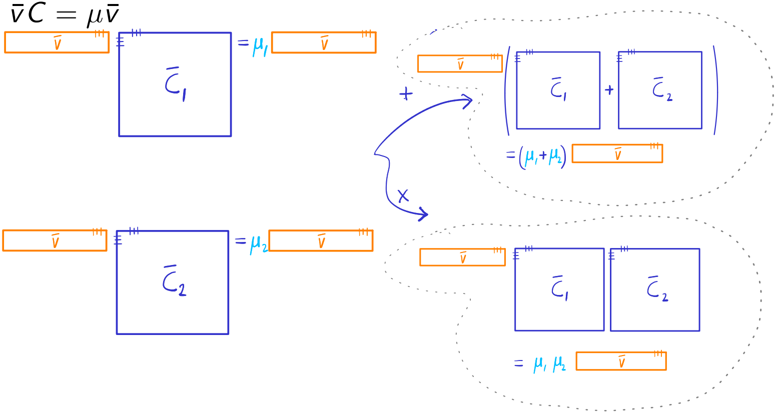

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$+$

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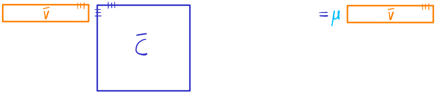
ELIMINATION

# How to Fix? *Approximate* Eigenvector Encryption

- $\bar{C}$ encrypts a bit $\mu$ under secret $\bar{v}$ if $\bar{v}\bar{C} + \bar{e} = \mu\bar{v}$ for "short" $\bar{e}$



- Do we have an FHE?
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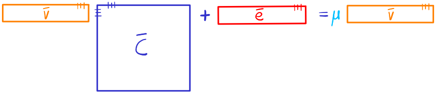

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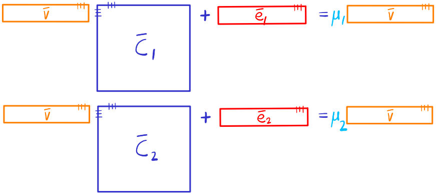

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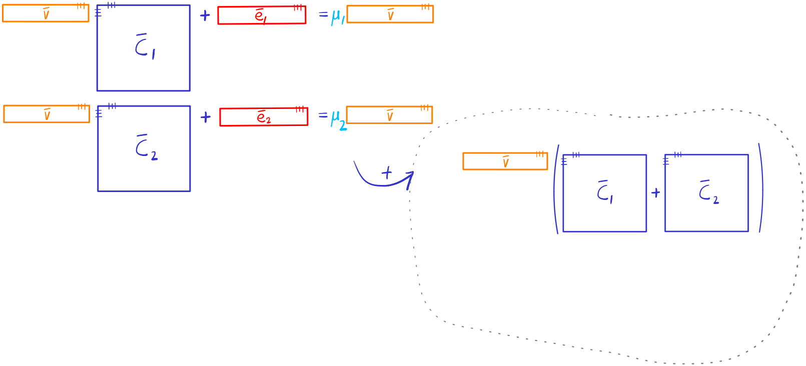

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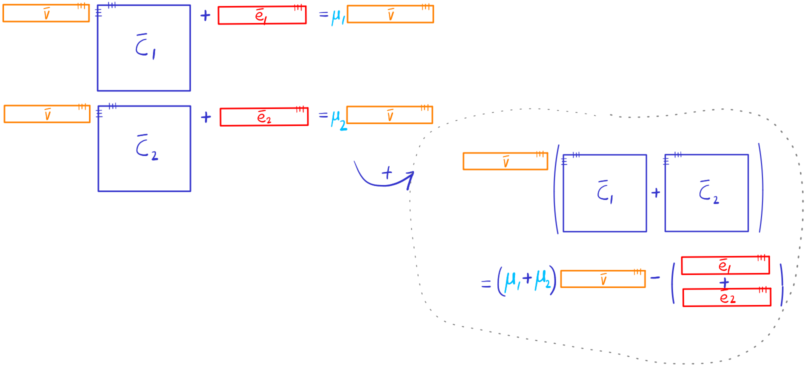

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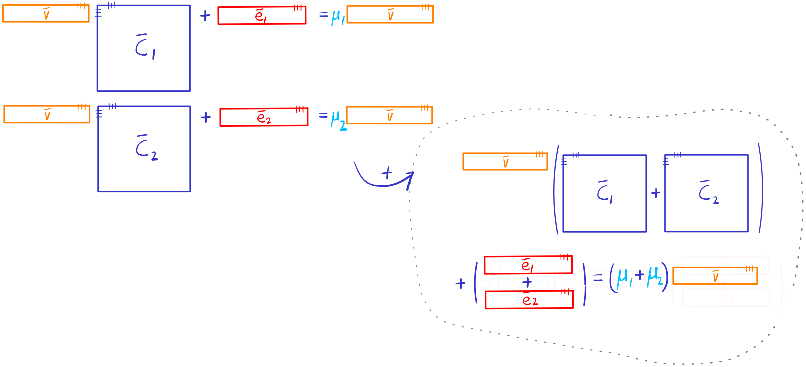

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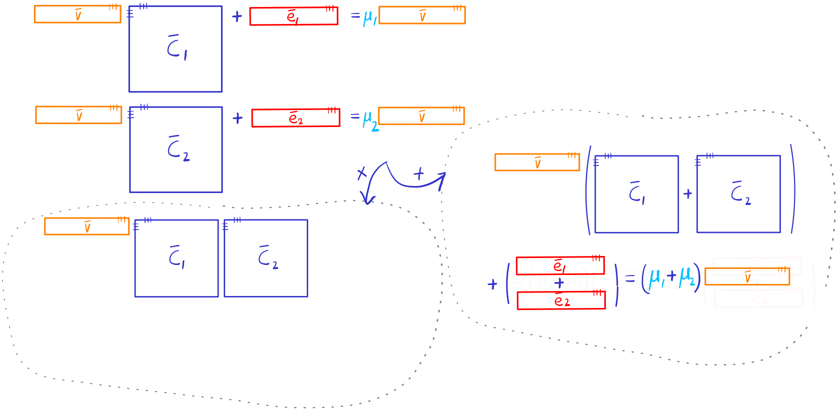

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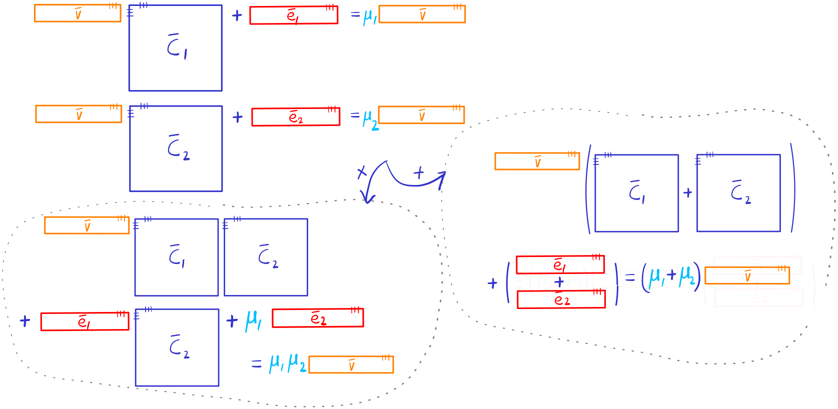

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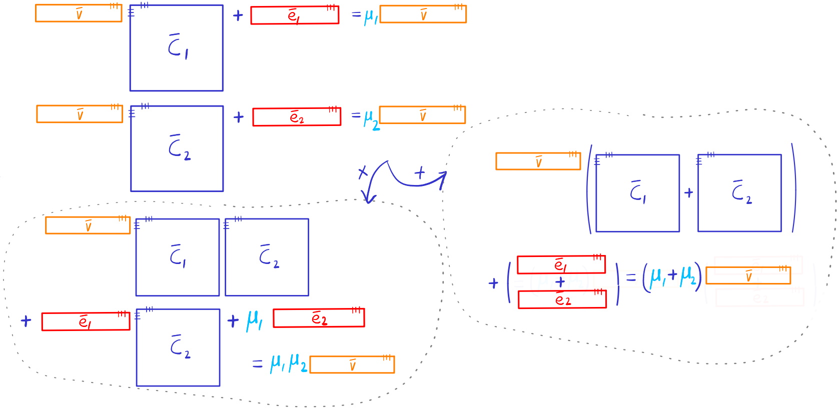
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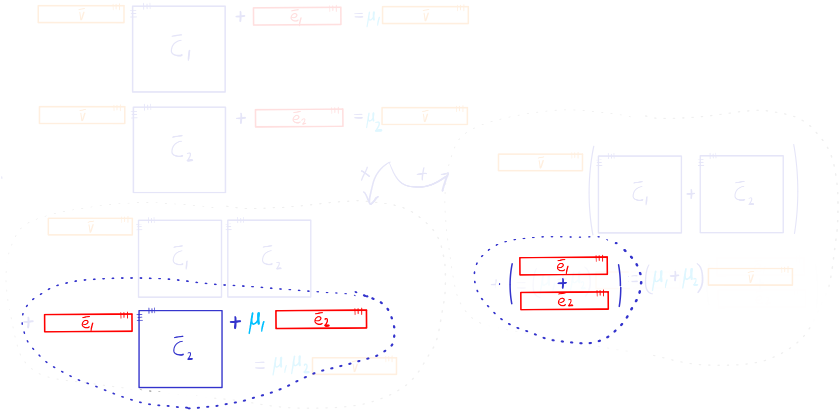

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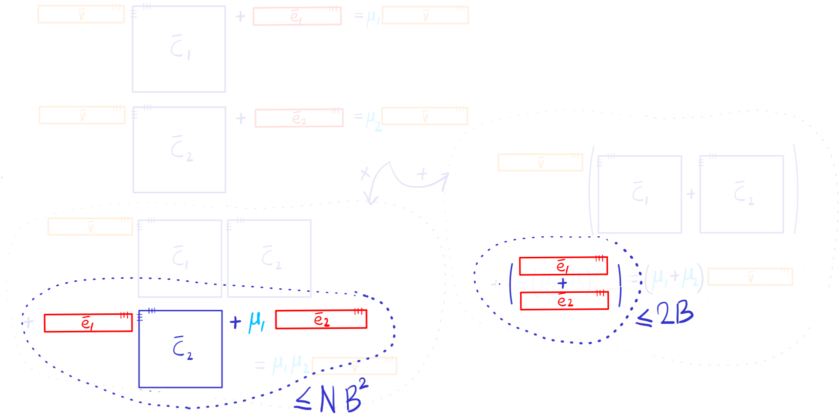

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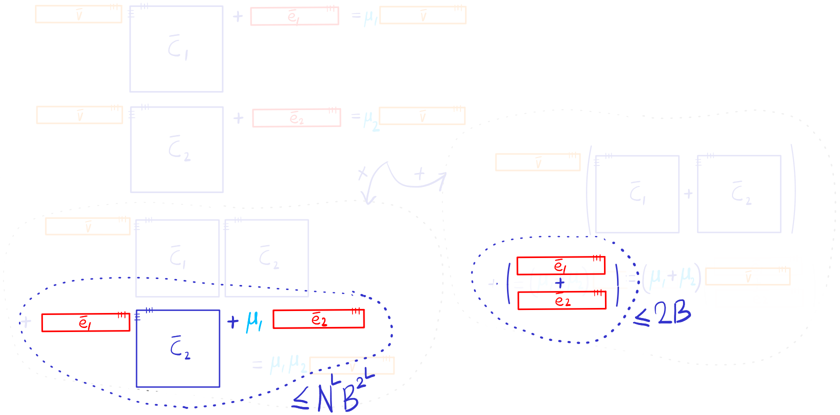

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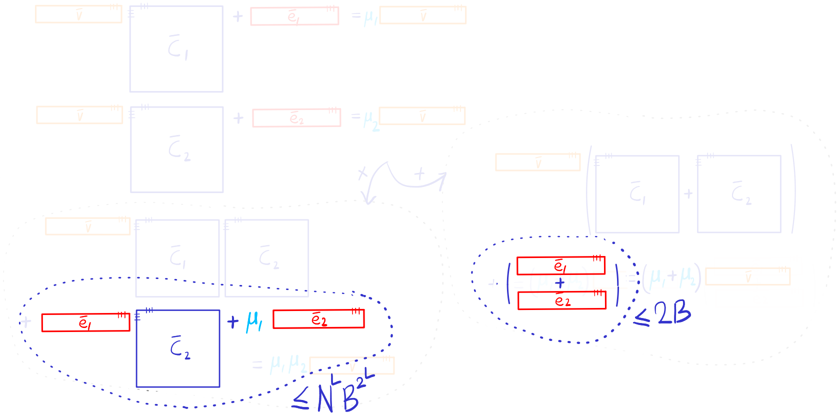

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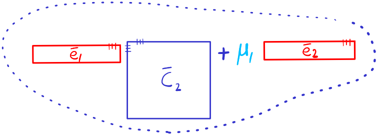
# Supporting Arbitrary Depth



- Two tricks:
  1. Stick to messages $\mu$ from $\{0, 1\}$ and $F$ with NAND gates
  2. "Flattening": embed matrix $\bar{C}$ into a higher dimensional matrix $\bar{C}'$ such that
     - 2.1 $\bar{C}'$ has low (infinity) norm
     - 2.2 Certain inner products "preserved"

     Implemented using "gadget" matrix $\bar{G} : \mathbb{Z}_q^{n\times N} \to \mathbb{Z}_q^{n\times m}$

     bit-decomposition function $G^{-1} : \mathbb{Z}_q^{n\times m} \times \mathbb{Z}_q^{n\times N}$

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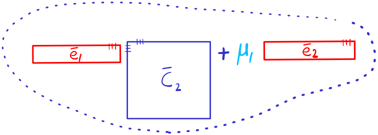

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$$\sum_{k\in[\ell]} a_{11k}\, 2^k = a_{11}$$

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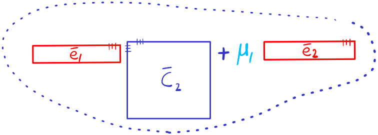

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$\boxed{a_{11} \cdots a_{1\ell}}$

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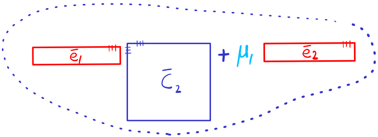

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$$\sum_{k\in[\ell]} a_{11k} 2^k = a_{11} \quad \circ\circ\circ \quad a_{1m} \qquad \boxed{a_{11} \circ\circ\circ\, a_{1\ell}} \quad \circ\circ\circ \quad \boxed{a_{n11} \circ\circ\circ\, a_{1m\ell}}$$

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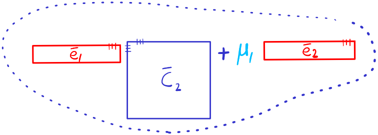

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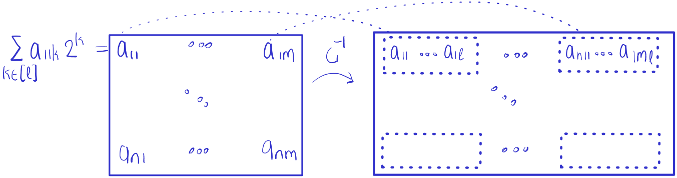

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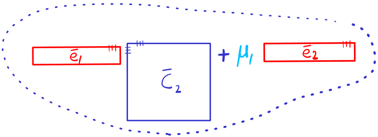

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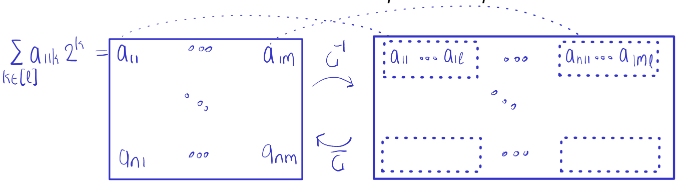

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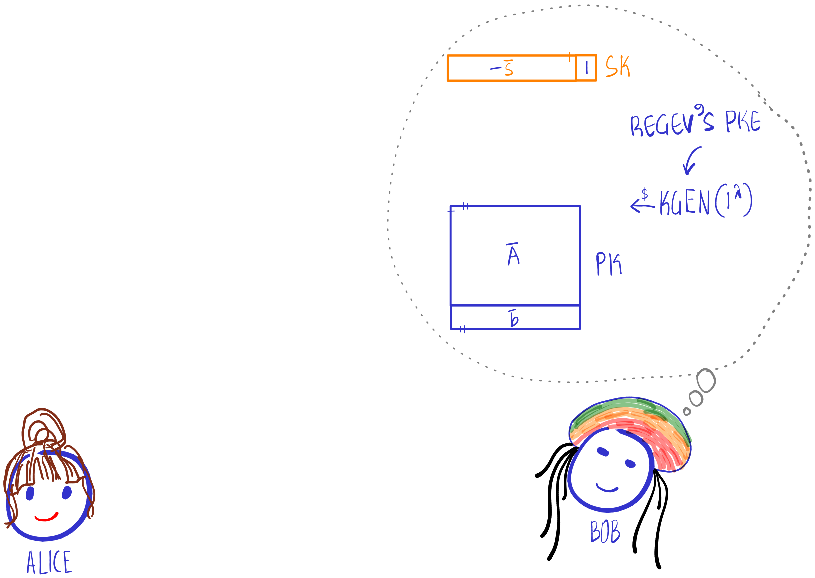

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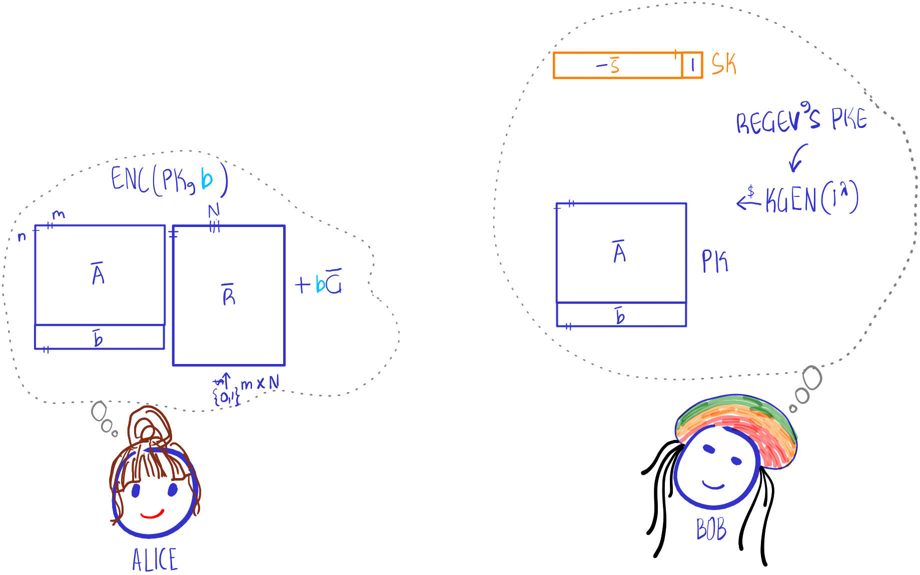

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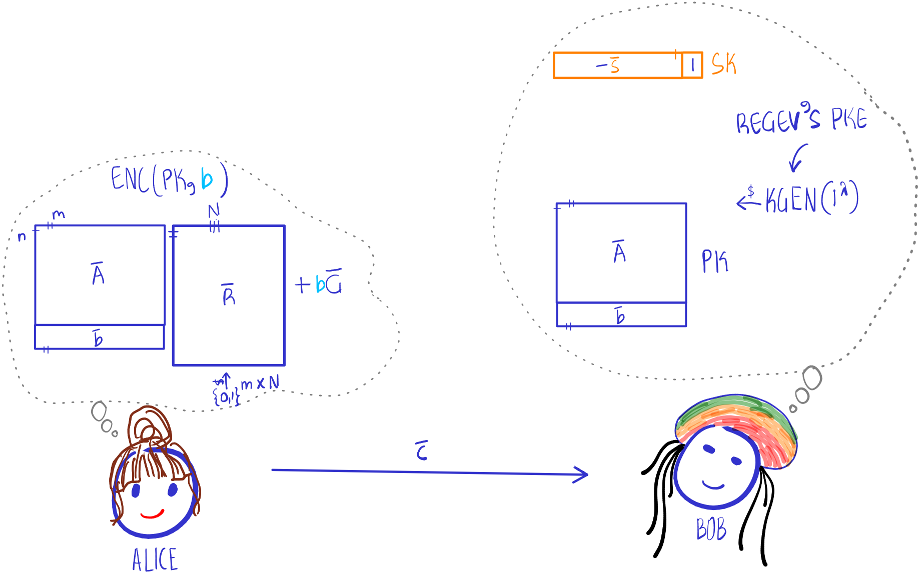

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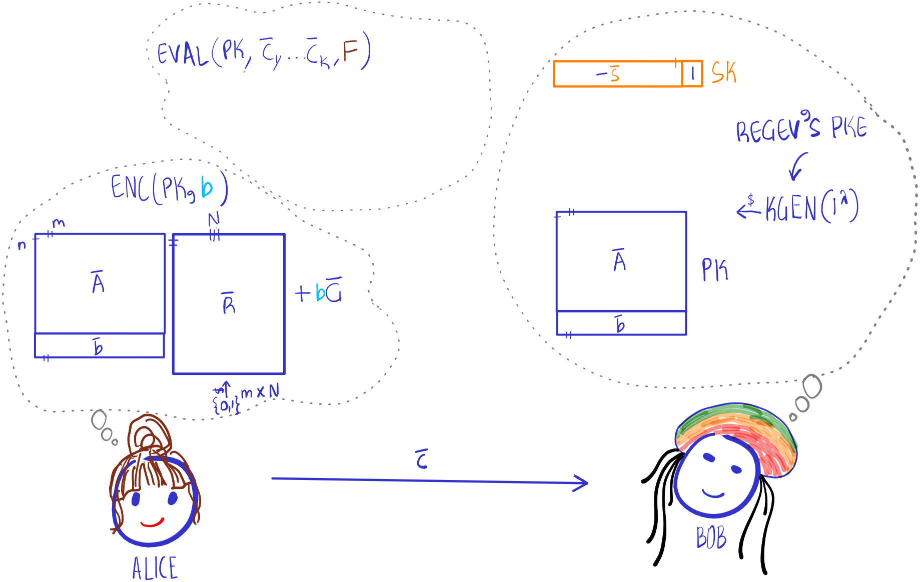

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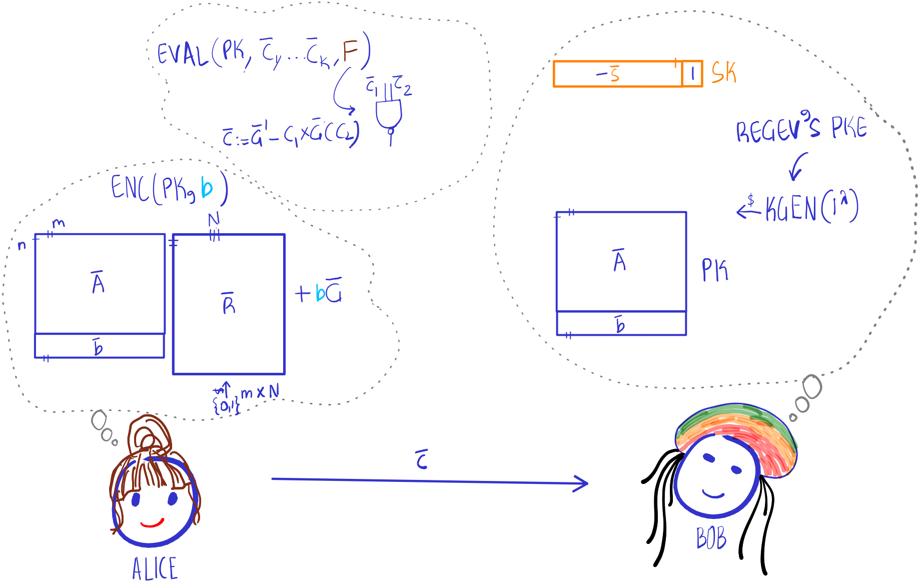


# Plan for this Session

Homomorphic Encryption

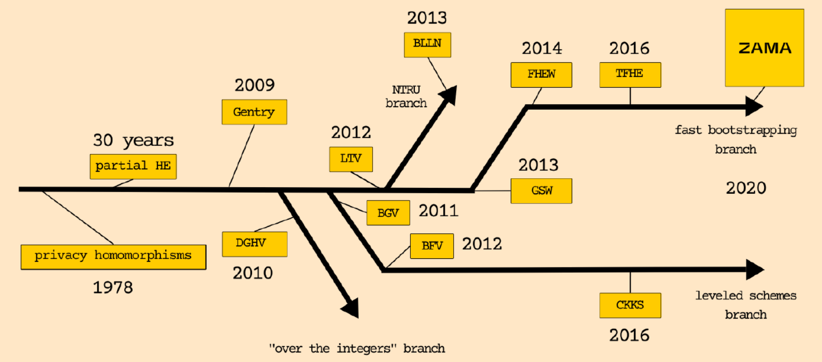
Fully-Homomorphic Encryption (FHE)

Learning with Errors (LWE)

Gentry-Sahai-Waters FHE from LWE

Wrapping Up

# Genealogy of FHE Schemes



COURTESY: ZAMA.AI

# To Recap

- Saw partially homomorphic encryption schemes
- Learned about LWE and Regev's PKE based on LWE
- GSW FHE via approximate eigenvectors

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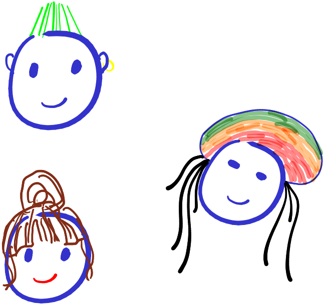
- Archisman's session for how to use FHE

# Thank You for Your Attention! Questions?



# References

1. The partially homomorphic schemes we discussed are from [ElG84, GM82]
2. The LWE problem was introduced in [Reg05], and the reduction from worst-case lattices problems was established in [Pei09]
3. The GSW FHE is from [GSW13]. The presentation here is from Halevi's survey [Hal17].
4. To learn more about lattices-based cryptography, the survey by Peikert [Pei16] is an excellent source.

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